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SUMS OF HEIGHTS OF ALGEBRAIC NUMBERS

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Abstract. For $A_t(x) = f(x) - t\,g(x)$, we consider the set $\{\sum_{A_t(\alpha)=0} h(\alpha) : t \in \overline{\mathbb{Q}}\}$. The polynomials $f(x), g(x)$ are in $\mathbb{Z}[x]$, with only mild restrictions, and $h(\alpha)$ is the Weil height of α. We show that this set is dense in $[d, \infty)$ for some effectively computable limit point d.

1. Introduction

Let α be an algebraic number, and $h(\alpha)$ its Weil height, as defined below. Much interest has recently been given to the spectrum of various modifications of this height operator h. For example, Schinzel [10] showed that the first positive number in the set $\Lambda_R = \{h(\alpha) : \alpha \text{ totally real}\}$ is $0.2406\ldots$. Smyth [11, 12] found the next three positive points in Λ_R and then demonstrated that Λ_R has a limit point of $0.27328\ldots$ and is dense beyond that point; Flammang [8] found the fifth and sixth points in this spectrum (actually, for totally positive algebraic numbers, but this amounts to the same thing). Similar results hold for $\Lambda_2 = \{h(\alpha)+h(1-\alpha) : \alpha \in \overline{\mathbb{Q}}\}$ (as seen in Zhang [15] and Zagier [14], with limit points and regions of density found by Doche [4, 3]) and for $\Lambda_3 = \{h(\alpha) + h(1 - \frac{1}{\alpha}) + h(\frac{1}{1-\alpha}) : \alpha \in \overline{\mathbb{Q}}\}$ (see Dresden [5]). In this article, we prove the following theorem, which generalizes some of the previously stated results (as defined later, ℓ and C are the leading coefficient and content of a polynomial).

Theorem 1.1. *Let $f(x), g(x)$ be fixed polynomials in $\mathbb{Z}[x]$ with $\deg f > \deg g \geq 0$, $\ell(f) \mid C(g)$, and $C(f) = 1$. For t any complex number, let $A_t(x) = f(x) - t\,g(x)$. Then, there exists an effectively computable limit point $d \geq 0$ (depending only on f and g) such that the set $\{\sum_{A_t(\alpha)=0} h(\alpha) \;:\; t \in \overline{\mathbb{Q}}\}$ is dense in $[d, \infty)$.*

As an aside, we note that we might have $f(x) = x$ and $g(x) = 1$, in which case the set in Theorem 1.1 is simply $\{h(\alpha) : \alpha \in \overline{\mathbb{Q}}\}$. This is trivially dense in $[0, \infty)$, as seen by considering $\alpha = p^{1/n}$ for p a prime number, but our methods in the proof of Theorem 1.1 would produce a d of only $0.2406\ldots$. In light of this, we must state that there is no guarantee that our constant d is the "best" constant. We also note that we can assume without loss of generality that $f(x)$ and $g(x)$ are relatively prime. If they share a common factor (say $q(x)$), then we could write $f(x) - t\,g(x) = (f_0(x) - t\,g_0(x)) \cdot q(x)$, and if we let $A_{0,t}(x) = f_0(x) - t\,g_0(x)$, then $\sum_{A_t(\alpha)=0} h(\alpha) = \sum_{A_{0,t}(\alpha)=0} h(\alpha) + \sum_{q(\alpha)=0} h(\alpha)$. That last sum will be some fixed

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number independent of t, and when added to the d (from f_0 and g_0) in Theorem 1.1 will produce the correct limit point for f and g.

With the help of Theorem 1.1, we can also prove

Corollary 1.2. *Let G be a finite group of fractional linear transforms $\sigma(x)$, where*

$$\sigma(x) = \frac{ax+b}{cx+d}, \qquad ad-bc \neq 0, \qquad a,b,c,d \in \mathbb{Q}.$$

Then, there exists an effectively computable limit point $e \geq 0$ (depending only on G) such that the set $\left\{\sum_{\sigma \in G} h(\sigma(\alpha)) : \alpha \in \overline{\mathbb{Q}}\right\}$ is dense in $[e, \infty)$.

Examples and applications of both Theorem 1.1 and Corollary 1.2 are given in the last section of this paper.

2. Definitions

Let α be an algebraic number, and let $p(x)$ be its minimal polynomial in $\mathbb{Z}[x]$,

$$p(x) = p_n x^n + \cdots + p_1 x + p_0 = p_n \prod_{i=1}^{n} (x - \alpha_i).$$

We define the Mahler measure $M(p)$ as $M(p) = |p_n| \prod_{i=1}^{n} \max(|\alpha_i|, 1)$. We also define $C(f)$ to be the content of a polynomial $f(x)$; that is, the gcd of its coefficients, and $\ell(f)$ to be the leading coefficient. We will use $\log^+ |z|$ to represent $\log\max(|z|, 1)$, allowing us to state

$$h(\alpha) = \frac{1}{\deg p} \left\{ \log|\ell(p)| + \sum_{p(\alpha_i)=0} \log^+ |\alpha_i| \right\} = \frac{1}{\deg p} \log M(p). \tag{2.1}$$

For $\sigma(x) = \frac{ax+b}{cx+d}$ as in Corollary 1.2, we can think of σ as a member of $PGL(2, \mathbb{Q})$ if we write $\sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, acting on a projective point $\begin{bmatrix} x \\ 1 \end{bmatrix}$. We will view σ in both ways in this article. Examples of such operators are $\sigma(x) = 1 - x$ (in which $\sigma^2(x) = x$) and $\sigma(x) = \frac{1}{1-x}$ (for which $\sigma^3(x) = x$). Finally, we will use $\Re(z)$ and $\Im(z)$ to denote the real and imaginary parts of z, and we recall that a totally real (totally positive) algebraic number is a number α such that it and all its conjugates are real (positive) numbers.

3. Polynomials

Throughout this section, $f(x)$ and $g(x)$ are as defined in Theorem 1.1.

Lemma 3.1. *Let T be a complete set of algebraic conjugates of some algebraic integer. Define $R(x) \in \mathbb{Z}[x]$ as*

$$R(x) = \prod_{t \in T} \{f(x) - t\, g(x)\},$$

where $f(x), g(x)$ are as defined above. Then, $C(R)$, the gcd of the coefficients of $R(x)$, is 1.

Proof. Let $n = \deg f$, and write $R(x)$ as

$$R(x) = \prod_{t\in T} \left\{ f_n x^n + \cdots + (f_k - tg_k)x^k + \cdots + (f_0 - tg_0) \right\}, \tag{3.1}$$

and suppose that p is a prime such that $p \,|\, C(R)$. Clearly, this implies $p \,|\, f_n$, and since $f_n = \ell(f)$, which divides $C(g)$, then $p \,|\, g_i \ \forall\ i$. Now since $C(f) = 1$, choose k such that $p \,|\, f_{k+i} \ \forall\ i > 0$ but $p \nmid f_k$, and consider the coefficient of $x^{k|T|}$ in $R(x)$, in (3.1) above. This coefficient is a sum of many terms, all but one of which contain a factor of the form $(f_{k+i} - t\, g_{k+i})$ with $i > 0$, and hence are divisible by p. The one term which does not contain such a factor is the term found by multiplying together all the $(f_k - tg_k)x^k$'s in (3.1) above, giving us

$$\prod_{t\in T} (f_k - tg_k)x^k = \left({f_k}^{|T|} - \left(\sum_{t\in T} t \right) {f_k}^{|T|-1} g_k + \text{other terms with } g_k \ \ldots \right) x^{k|T|}.$$

Recalling that $p \,|\, g_i \ \forall\ i$ but $p \nmid f_k$, we can conclude that p does not divide this one (large) term, and hence does not divide the entire coefficient of $x^{k|T|}$. Thus, $p \nmid C(R)$. □

In addition to the coefficients of this polynomial $R(x) \in \mathbb{Z}[x]$, we are also interested in its roots. The following lemma gives us some useful information that we will need in the sequel:

Lemma 3.2. *For $f(x), g(x)$ as above, let $\alpha_1, \ldots \alpha_{\deg g}$ represent the roots of $g(x)$ (with possible repetition), and let t be any real number. Then, for all sufficiently large $|t|$, we can label the roots of $f(x) - t\, g(x) = 0$ as $\alpha_{1,t}, \ldots \alpha_{\deg f,t}$ such that*

$$\lim_{|t|\to\infty} \alpha_{j,t} = \begin{cases} \alpha_j & j \le \deg g, \\ \infty & j > \deg g. \end{cases}$$

Furthermore, for α_j a root of g on the unit circle, and for all $|t|$ sufficiently large and of the same sign, then the root $\alpha_{j,t}$ either always is on the unit circle, or never is.

Proof. We first point out that (for $t \neq 0$) the roots of $f(x) - t\, g(x)$ are exactly the roots of $g(x) - f(x)/t$, and that the latter function converges uniformly (in t) on bounded subsets of $\mathbb{C}$ to $g(x)$. With this in mind, consider a disc with radius A such that all the roots of $g(x)$ are contained in its interior. Since $g(x)$ and $f(x)/t$ are analytic on this disc and since $f(x)/t$ converges to 0 as $|t| \to \infty$, we can apply Rouché's Theorem [13, p. 22] to assert that for all $|t|$ sufficiently large, $f(x) - t\, g(x)$ has exactly $\deg g$ roots inside this disc of radius A. Let us label the roots outside the disc as $\alpha_{j,t}$ for j from $1 + \deg g$ to $\deg f$ (we know there is at least one such root, as $\deg g < \deg f$). Since A can be chosen to be arbitrarily large, we can conclude that $\lim_{|t|\to\infty} \alpha_{j,t} = \infty$ for $j > \deg g$. Returning to the roots inside the circle, let us draw a small disc around each root α_j of $g(x)$ such that no other distinct root of g is contained therein. By Rouché's Theorem, we can (for $|t|$ sufficiently large) label our roots of $f(x) - t\, g(x)$ as $\alpha_{j,t}$ such that each $\alpha_{j,t}$ is contained in the small disc about α_j. These $\alpha_{j,t}$ are continuous functions of t, so the only possible ambiguity in labelling them would occur if one of the α_j's has multiplicity $n > 1$. In this case, Rouché's Theorem tells us we have n of the $\alpha_{j,t}$'s in our small disc around α_j, and we simply need a consistent labelling scheme (say, by magnitude and then by argument) to avoid confusion. Taking smaller and smaller discs as $|t| \to \infty$, we can conclude that $\lim_{|t|\to\infty} \alpha_{j,t} = \alpha_j$ for $j \le \deg g$.

Let us now prove the second part of this lemma. Suppose $\alpha_j = e^{i\theta_0}$ is a root of g on the unit circle. If, for some particular t of sufficiently large magnitude, $f(x) - t\, g(x)$ has a root $\alpha_{j,t} = e^{i\theta}$ on the unit circle and close to $e^{i\theta_0}$, then we have

$$f(\cos\theta + i\sin\theta) - t\, g(\cos\theta + i\sin\theta) = 0, \tag{3.2}$$

or, put in another way,

$$\Im\left(\frac{f(\cos\theta + i\sin\theta)}{g(\cos\theta + i\sin\theta)}\right) = \Im(t) = 0, \tag{3.3}$$

which implies

$$\Im\left(f(\cos\theta + i\sin\theta)\cdot g(\cos\theta - i\sin\theta)\right) = 0. \tag{3.4}$$

For θ near θ_0, this trigonometric polynomial either is identically zero or is zero only at $\theta = \theta_0$; this implies that for all t of sufficiently large magnitude this root $\alpha_{j,t}$ mentioned above either always is on the unit circle (near $e^{i\theta_0}$) or never is on the circle (but still approaches $e^{i\theta_0}$). This completes the proof. □

For $f(x)$, $g(x)$ as above, let us define $m(t)$ as

$$m(t) = \prod_{A_t(\alpha)=0} \max\{1, |\alpha|\}, \qquad \text{where } A_t(x) = f(x) - t\, g(x). \tag{3.5}$$

It should be clear that $m(t)$ is continuous, as the roots of $f(x) - t\, g(x)$ are continuous functions of their coefficients, which are themselves all continuous in t. We now see how this function $m(t)$ is related to the sum of heights.

Proposition 3.3. *Let $f(x)$, $g(x)$ be as in Theorem 1.1. Let T be a complete set of algebraic conjugates of some totally real algebraic integer t, and as before let $A_t(x) = f(x) - t\, g(x)$. Then,*

$$\sum_{A_t(\alpha)=0} h(\alpha) = \log \ell(f) + \frac{1}{|T|}\sum_{t\in T} \log m(t). \tag{3.6}$$

Proof. To find the heights of the roots of $A_t(x) = f(x) - t\, g(x)$, we must first find the minimal polynomials of those roots. Let us suppose that our polynomial $A_t(x)$ factors over $\mathbb{Q}(t)$ as

$$A_t(x) = f(x) - t\, g(x) = \prod_{i=1}^{I} p_i(x;t), \tag{3.7}$$

for each $p_i(x;t)$ irreducible and in $\mathbb{Q}(t)[x]$. Define $P_i(x) = \prod_{t\in T} p_i(x;t)$, a polynomial in $\mathbb{Q}[x]$, and note that $\prod_{i=1}^{I} P_i(x) = \prod_{t\in T} f(x) - t\, g(x) = R(x)$, where $R(x)$ (as seen in Lemma 3.1) is in $\mathbb{Z}[x]$ with $C(R) = 1$. Hence, we can assume without loss of generality (by multiplying the polynomials $p_i(x;t)$ by appropriate constants) that the polynomials $P_i(x)$ are in $\mathbb{Z}[x]$. Furthermore, since $C(R) = 1$, then $C(P_i) = 1$ for every i.

Let α be a root of $A_t(x)$. Then, α is a root of $p_i(x;t)$ for some i, and hence of P_i. Since P_i has degree $\deg p_i(x;t)\cdot|T|$, and α is of degree

$$[\mathbb{Q}(\alpha):\mathbb{Q}(t)]\cdot[\mathbb{Q}(t):\mathbb{Q}] = \deg p_i(x;t)\cdot|T|,$$

and since $P_i \in \mathbb{Z}[x]$ with $C(P_i) = 1$, then P_i is the minimal polynomial for α.

So, recalling the connection between heights and Mahler measure in equation (2.1), we have

$$\begin{aligned} h(\alpha) &= \frac{1}{\deg P_i(x)} \log M(P_i) \\ &= \frac{1}{|T| \cdot \deg p_i(x;t)} \left(\log \ell(P_i) + \sum_{P_i(\gamma)=0} \log^+ |\gamma| \right). \end{aligned}$$

This holds for every root of $p_i(x;t)$, so summing over these roots we get

$$\sum_{p_i(\alpha;t)=0} h(\alpha) = \frac{1}{|T|} \left(\log \ell(P_i) + \sum_{P_i(\gamma)=0} \log^+ |\gamma| \right).$$

We now sum over i, and since $\prod_i p_i(x;t) = A_t(x)$, our formula becomes

$$\begin{aligned} \sum_{A_t(\alpha)=0} h(\alpha) &= \frac{1}{|T|} \sum_{i=1}^{I} \left(\log \ell(P_i) + \sum_{P_i(\gamma)=0} \log^+ |\gamma| \right) \\ &= \frac{1}{|T|} \left(\log \ell(R) + \sum_{i=1}^{I} \sum_{P_i(\gamma)=0} \log^+ |\gamma| \right). \end{aligned}$$

Let us now consider that last double sum, which is over all the roots of $R(x) = \prod_{i=1}^{I} P_i(x)$. Since we also have $R(x) = \prod_{t \in T} f(x) - t\, g(x) = \prod_{t \in T} A_t(x)$, we can rewrite the above equation as

$$\sum_{A_t(\alpha)=0} h(\alpha) = \frac{1}{|T|} \log \ell(R) + \frac{1}{|T|} \sum_{t \in T} \sum_{A_t(\gamma)=0} \log^+ |\gamma|.$$

Using equation (3.5) and the definition of $R(x)$, we have our result:

$$\sum_{A_t(\alpha)=0} h(\alpha) = \log \ell(f) + \frac{1}{|T|} \sum_{t \in T} \log m(t). \qquad \square$$

4. Sets of algebraic numbers

Our goal in this section is to generalize certain results of Smyth's on algebraic numbers. In particular, we want to find sets of totally real algebraic numbers $B_n = \{\beta_{n,i}\}_{i=1}^{2^n}$ such that

$$\text{(4.1)} \qquad \text{the limit} \qquad \mathcal{L}_0 = \lim_{n \to \infty} \frac{1}{|B_n|} \sum_{\beta \in B_n} \log m(\beta) \qquad \text{exists.}$$

This will be a major step in establishing the existence of a limit point for the set $\left\{ \sum_{A_t(\alpha)=0} h(\alpha) \right\}$, as can be gleaned from formula (3.6). We then want to find other sets of algebraic numbers B_n such that the limits in equation (4.1) will be dense beyond $\mathcal{L}_0$. Fortunately, much of this work has been done by C. Smyth in [11]. (Smyth's totally real algebraic numbers β_n have proved to be quite useful. It's not hard to show that $(\beta_n)^2$ is a totally *positive* algebraic number, and a complete set of algebraic conjugates of $(\beta_n)^2$ is simply the set $\{(\beta_{n,i})^2 \ : \ \beta_{n,i} \in B_n\}$. These sets are used by V. Flammang [7] on lengths of polynomials, and by M. J. Bertin [1] on lower bounds for the heights of totally positive algebraic integers.)

We begin with b any odd positive integer. Define $\beta_0 = b$, and $\beta_n > 0$ by $H(\beta_n) = \beta_{n-1}$ $(n = 1, 2, \dots)$, where $H(x) = x - \dfrac{1}{x}$. We have

Theorem 4.1 (Smyth, 1980)**.** *β_n (defined above) is a totally real algebraic integer of degree 2^n over the rationals. In the case where $\beta_0 = b = 1$, the sequence*

$$h(\beta_1),\ h(\beta_2),\ h(\beta_3),\ h(\beta_4),\ \cdots \approx 0.241\dots,\ 0.261\dots,\ 0.268\dots\ 0.272\dots,$$

of elements of Λ_R has limit point $0.27328\dots$. Furthermore, Λ_R is dense beyond this point.

Proof. Theorems 1 and 2, and Lemmas 4 and 5, in [11]. □

If we let the set $B_n = \{\beta_{n,i}\}_{i=1}^{2^n}$ consist of β_n and all its conjugates, then we can define the distribution function $F_n(x)$ as

$$F_n(x) = \frac{1}{2^n} \cdot (\text{the number of elements of } B_n \text{ in } (-\infty, x])\,.$$

We note that since $dF_n(x)$ is a discrete measure supported only on the set B_n, then

$$\int_{-\infty}^{\infty} \phi(x) dF_n(x) = \frac{1}{2^n} \sum_{i=1}^{2^n} \phi(\beta_{n,i}). \tag{4.2}$$

With this in mind, we can prove

Theorem 4.2 (Smyth, 1980)**.** *Let $F_n(x)$ be as above (with $\beta_0 = b$ any odd positive integer), let $\mu(x) \geq 1$ be a continuous function, and let m_1, m_2, and A be positive integers. Then*

(1) *$F(x) = \lim_{n\to\infty} F_n(x)$ exists and is a continuous function.*

(2) *If $\mu(x) \leq (m_1)|x|^{m_2}$ for $|x| > A$, and if $\mu(x)$ is monotone decreasing on $(-\infty, -A)$ and monotone increasing on (A, ∞), then*

$$\mathcal{L}_0 = \lim_{n\to\infty} \int_{-\infty}^{\infty} \log \mu(x) dF_n(x) \quad \textit{exists, and is equal to} \quad \int_{-\infty}^{\infty} \log \mu(x) dF(x).$$

(3) *If, in addition, $(1/m_1)|x|^{m_2} \leq \mu(x)$, then for $\omega > 0$, $\epsilon > 0$ arbitrarily chosen, there exist a positive integer $n > 0$ and a positive odd integer $b = \beta_0$ such that*

$$\left| \frac{1}{2^n} \sum_{i=1}^{2^n} \log \mu(\beta_{n,i}) \ - \ (\mathcal{L}_0 + \omega) \right| < \epsilon$$

(where the numbers $\beta_{n,i}$ are as defined above). In other words, we can make $\int_{-\infty}^{\infty} \log \mu(x) dF_n(x)$ arbitrarily close to any number greater than $\mathcal{L}_0$.

Proof. This is a generalization of Lemma 9 and Theorem 2 in Smyth's paper [11]. A complete proof is given in Theorems 6.2.10 and 7.3.1 in [5]. □

5. Limit points and density

At this point, we need only show that the function $m(t)$ defined in (3.5) satisfies the conditions on $\mu(t)$ as listed in Theorem 4.2. This will enable us to prove Theorem 1.1.

5.1. **Results on** $m(t)$**.**

Lemma 5.1. *For* $f(x)$*,* $g(x)$ *as in Theorem 1.1, a function* $m(t)$ *as in* (3.5) *satisfies the conditions of Theorem 4.2. In other words, there is some positive number* A *such that*

(1) *there exist positive integers* m_1 *and* m_2 *such that* $m(t)$ *satisfies*
$$\frac{1}{m_1}|t|^{m_2} \;\le m(t) \le\; m_1|t|^{m_2} \qquad \textit{for } |t| > A.$$
(2) $m(t)$ *is monotone decreasing on* $(-\infty, -A)$ *and monotone increasing on* (A, ∞)*.*

Proof. If f, g satisfy the conditions of Theorem 1.1, then so do $f, -g$, so we can assume without loss of generality that t is positive.

We recall Jensen's formula, which states that for F holomorphic, $F(0) \neq 0$, $r > 0$, and $\alpha_1, \dots, \alpha_N$ the zeroes of F in the closed disc $\overline{D}(0, r)$ listed with multiplicities, we have

$$|F(0)| \prod_{j=1}^{N} \frac{r}{|\alpha_j|} = \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |F(re^{i\theta})| d\theta \right\}. \tag{5.1}$$

In particular, for $P(x)$ a polynomial with roots α_j and leading coefficient $\ell(P)$, with $P(0) \neq 0$, since

$$|P(0)| = |\ell(P)| \prod_{|\alpha_j| \le r} |\alpha_j| \prod_{|\alpha_j| > r} |\alpha_j|, \tag{5.2}$$

then Jensen's formula gives us

$$|\ell(P)| \prod_{|\alpha_j| > r} |\alpha_j| = \frac{1}{r^N} \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |P(re^{i\theta})| d\theta \right\}, \tag{5.3}$$

where N is the number of roots of $P(x)$ that have absolute value $\le r$.

As mentioned earlier, we will use $\Re(z)$ for the real part of a complex expression. Recall that $-|z| \le \Re(z) \le |z|$. We should also point out that $f(0)$ and $g(0)$ cannot both be zero (as f, g are, without loss of generality, assumed to be relatively prime polynomials; see the note after Theorem 1.1) and so $f(0) - t\,g(0)$ is zero for at most one value of t.

We now prove part (1). Choose R such that all roots of $g(x) = 0$ are inside $D(0, R/2)$. By Lemma 3.2, we know that for t sufficiently large, the roots of $f(x) - t\,g(x) = 0$ are all within ϵ of the roots of $g(x) = 0$ or outside a circle of radius $2R$. In particular, $\left|\frac{f(x)}{t} - g(x)\right|$ has no zeroes on the circle of radius R, and in fact, is both bounded and bounded away from 0 on this circle. We also know that $\prod_{R > |\alpha_{j,t}| > 1} |\alpha_{j,t}|$ is a bounded number.

Thus, by the Dominated Convergence Theorem we have:

$$\lim_{t\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left| \frac{f(Re^{i\theta})}{t} - g(Re^{i\theta}) \right| d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left| g(Re^{i\theta}) \right| d\theta. \tag{5.4}$$

Now we apply (5.3) with $P(x) = \dfrac{f(x)}{t} - g(x)$ and $r = R$, to find that

$$\frac{|\ell(f)|}{t} \cdot \prod_{|\alpha_{j,t}| > R} |\alpha_{j,t}| = \frac{1}{R^N} \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left| \frac{f(Re^{i\theta})}{t} - g(Re^{i\theta}) \right| d\theta \right\}, \tag{5.5}$$

where $\ell(f)$ is the leading coefficient of f. As $t \to \infty$, by (5.4) the right-hand side of (5.5) converges to

$$\frac{1}{R^N} \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|g(Re^{i\theta})\right| d\theta\right\},$$

which (since g is nonzero on this circle) is well defined and equal to some constant, g_0. So, for all t sufficiently large,

$$\left|\frac{|\ell(f)|}{t} \cdot \prod_{|\alpha_{j,t}|>R} |\alpha_{j,t}| \quad - \quad g_0\right| < \epsilon.$$

(We note that each root $\alpha_{j,t}$ of $f(x) - t\,g(x)$ is a function of t.) So, for all t sufficiently large,

$$\left|\frac{|\ell(f)|}{t} \cdot \prod_{|\alpha_{j,t}|>1} |\alpha_{j,t}| \quad - \quad g_0 \prod_{R>|\alpha_{j,t}|>1} |\alpha_{j,t}|\right| < \left(\epsilon \cdot \prod_{R>|\alpha_{j,t}|>1} |\alpha_{j,t}|\right) < \epsilon'.$$

So, we get

$$\left|\frac{m(t)}{t} \quad - \quad \frac{g_0}{|\ell(f)|} \prod_{R>|\alpha_{j,t}|>1} |\alpha_{j,t}|\right| < \frac{\epsilon'}{\ell(f)} = \epsilon''.$$

As t grows, the roots $\alpha_{j,t}$ on the left of the above equation approach the roots of g, and so for t sufficiently large, there exists an $m_1 > 0$ such that

$$\frac{1}{m_1} < \frac{m(t)}{t} < m_1,$$

and so

$$\frac{1}{m_1} t < m(t) < m_1 t.$$

With $m_2 = 1$, this satisfies part (1).

Now we will prove part (2). Recall that we need only concern ourselves with t positive.

First suppose g has no roots on the unit circle. Since the roots of $f(x) - t\,g(x)$ approach ∞ and the roots of g, then for all sufficiently large t we can assume that the roots of $f(x) - t\,g(x)$ are bounded away from the unit circle. In particular, $1/|f - tg|$ is bounded on the unit circle (as is, of course, $|f - tg|$ itself). Taking $r = 1$ in (5.3), we see that

$$m(t) = \frac{1}{|\ell(f)|} \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|f(e^{i\theta}) - t\,g(e^{i\theta})\right| d\theta\right\}. \tag{5.6}$$

By our comments above on $|f - tg|$, we can take the derivative of the above equation with respect to t, and find that

$$\begin{aligned}
\frac{d}{dt} m(t) &= m(t) \cdot \frac{d}{dt}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f - tg|\, d\theta\right\} \\
&= m(t) \cdot \frac{d}{dt}\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi} \log\left[(f - tg)(\overline{f} - t\overline{g})\right] d\theta\right\} \\
&= m(t) \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{t|g|^2 - \Re\{\overline{g}f\}}{|f - tg|^2} d\theta. \qquad (5.7)
\end{aligned}$$

This denominator $|f - tg|^2$ is always positive (and is nonzero on the circle of radius one).

For $t > \sup_{|z|=1} \dfrac{|f|}{|\overline{g}|}$ (well-defined since $g \neq 0$ on $|z| = 1$), the numerator of the integrand in (5.7) satisfies

$$t|g|^2 - \Re\{\overline{g}f\} \geq t|\overline{g}|^2 - |f||\overline{g}| > 0. \tag{5.8}$$

So, $\frac{d}{dt}m(t) > 0$, and $m(t)$ is monotone increasing.

Now suppose g does have roots on the unit circle. Choose t sufficiently large that, by Lemma 3.2, any of the roots $\alpha_{i,t}$ of $f - tg$ that will be on the unit circle are already there. Then, there is an $\epsilon > 0$ such that for all t' in a neighborhood of t the roots $\alpha_{i,t'}$ of $f - t'g$ are either in $\overline{D}(0,1)$ or outside $D(0, 1 + 2\epsilon)$, but not in the annulus in between. (The roots $\alpha_{i,t'}$ are continuous in t' for sufficiently small neighborhoods around t.) We then take $r = 1 + \epsilon$, and we note that

$$m(t) = \prod_{|\alpha_i|>1} |\alpha_i| = \prod_{|\alpha_i|>r} |\alpha_i|, \tag{5.9}$$

which by Jensen's formula (5.3) is

$$m(t) = \frac{1}{|\ell(f)|}\frac{1}{r^N} \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|f(re^{i\theta}) - t\,g(re^{i\theta})\right| d\theta\right\}, \tag{5.10}$$

and since $f - tg$ is both bounded and bounded away from zero on the circle of radius r, and g itself is nonzero as well, then we proceed as in (5.7) and (5.8), taking the derivative of $m(t)$ with respect to t to find that it is monotone increasing for t sufficiently large.

□

5.2. **Proof of Theorem 1.1.** We are now able to prove our original theorem.

Proof. Let $\beta_0 = b$ be any odd integer, and let the sets $B_n = \{\beta_{n,i}\}_{i=1}^{2^n}$ be as defined above. Let t be some $\beta_{n,i}$ in B_n, so that $A_t(x) = f(x) - \beta_{n,i}\,g(x)$. Then, using $T = B_n$, we can apply Proposition 3.3 to state

$$\sum_{A_t(\alpha)=0} h(\alpha) = \log \ell(f) + \frac{1}{2^n}\sum_{i=1}^{2^n} \log m(\beta_{n,i}). \tag{5.11}$$

We know from Lemma 5.1 that $m(t)$ satisfies the conditions of Theorem 4.2. We now use equation (4.2) and Theorem 4.2, part (2), to state

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} \log m(x)\, dF_n(x) \quad \text{exists, and equals} \quad \int_{-\infty}^{\infty} \log m(x)\, dF(x).$$

As before, we let $\mathcal{L}_0$ represent this limit. Then by part (3) of Theorem 4.2, for all $\omega > 0$, there exist $n > 0$ and an odd integer $b > 0$ such that

$$\left|\frac{1}{2^n}\sum_{i=1}^{2^n} \log m(\beta_{n,i}) - (\mathcal{L}_0 + \omega)\right| < \epsilon. \tag{5.12}$$

We use equation (5.11) on the above formula to give us

$$\left|\sum_{A_t(\alpha)=0} h(\alpha) - (\mathcal{L}_0 + \log \ell(f) + \omega)\right| < \epsilon. \tag{5.13}$$

Since ω was arbitrary, our sum is indeed dense in $[\mathcal{L}_0 + \log \ell(f), \infty)$. Let $d = \mathcal{L}_0 + \log \ell(f)$, and we get our effectively computable limit point. (When actually computing $\mathcal{L}_0$, we prefer to use equation (4.1)). $\square$

6. Proof of Corollary 1.2

For G a finite group as described in Corollary 1.2, we can find polynomials f, g that satisfy the conditions of Theorem 1.1 and such that the roots of $f(x) - t\,g(x) = 0$ are exactly of the form $\sigma(\alpha)$ as σ runs through G.

Our first theorem helps us establish the existence of these polynomials:

Theorem 6.1 (Vaaler)**.** *For G a finite subgroup of $PGL(2, K)$, there exist relatively prime monic polynomials f, g in $K[x]$, $0 \leq \deg(g) < \deg(f) = |G|$, such that for all t, the roots of $f(x) - t\,g(x) = 0$ are of the form $\sigma(\alpha)$ for $\sigma \in G$*

Remark. A more detailed version of this theorem, with a proof via Luroth's Theorem, first appeared in an early draft of [2].

Proof. Let $H \leq G$ be the subgroup of all affine elements $ax+b$ of G (note that since the identity element $1x + 0$ is in H, then H is nonempty). Let $s_1, \dots s_J$ be right coset representatives for H in G (with s_1, the coset representative for H itself, being $s_1(x) = x$), so that any $\sigma \in G$ can be written as $\sigma \in Hs_j$ or, more appropriately, $\sigma(x) = h_i(s_j(x))$. We note that if two elements of G have the same denominator $cx + d$, then they must be in the same coset Hs_j, as, given $\sigma_1(x) = \dfrac{ax+b}{cx+d}$ and $\sigma_2(x) = \dfrac{a'x+b'}{cx+d}$, then $\sigma_1 \cdot \sigma_2^{-1}$ is easily shown to be affine. However, if a third element σ_3 has $cx+d$ as a *numerator*, then $\sigma_1 \cdot \sigma_3^{-1}$ is easily shown to be nonaffine, and hence σ_3 is not in the same coset as σ_1 and σ_2.

Let $A(x) = \prod_H a_i x + b_i$, where the product is taken over all elements in H, and let $g(x) = \prod_{G-H} x + d_i/c_i$, where the product is taken over all elements $\dfrac{a_i x + b_i}{c_i x + d_i}$ in $G - H$ (if no such elements exist, we define $g(x) = 1$). Note that this product is well-defined, and makes $g(x)$ a monic polynomial. Finally, we define $f(x)$ as $f(x) = f_0 \cdot g(x) \cdot \sum_{j=1}^{J} A(s_j(x))$, with f_0 an appropriate constant to make $f(x)$ monic. Note that each $A(s_j(x))$ is simply the product of all elements in the coset Hs_j. Note also that $f(x)$ is a polynomial (since the $g(x)$ in front of the sum will cancel out all denominators in each $A(s_j(x))$). As for the degree of f, since $s_1(x) = x$, then $\deg A(x) = \deg A(s_1(x)) \geq \deg A(s_j(x))$, and so $\deg f(x) = \deg g(x) + \deg A(s_1(x)) = |G - H| + |H| = |G|$.

We now show that f and g are relatively prime. Suppose $g(x_0) = 0$; this implies $x_0 = -d/c$ for c, d from some term $\dfrac{ax+b}{cx+d}$ of G. Let $s_k(x)$ be the coset representative for the coset of H containing this term. (By our discussion above, while there may be more than one term with the denominator $cx + d$, they must all be in this same coset.) Now, let us consider what happens in our definition of $f(x)$ when we multiply $g(x)$ through the sum $\sum_j A(s_j(x))$. For $j = k$, all the $x - d/c$ factors in $g(x)$ will cancel with the corresponding denominators in $A(s_k(x))$. Furthermore, by our discussion above, $A(s_k(x))$ cannot have terms with numerator $cx + d$ (or a multiple thereof). Thus, $g(x)A(s_k(x))$ is not zero at $x = x_0$. However, for $j \neq k$, there is no such cancelling of terms, and so $g(x)A(s_j(x))$ is zero at $x = x_0$. As a result, $f(x_0) \neq 0$, and so $f(x), g(x)$ are relatively prime.

Finally, we consider the roots of $f(x) - t\,g(x) = 0$. Take $\sigma \in G$, and write $\sigma(x) = h_i(s_j(x))$, with notation as above. Then,

$$\begin{aligned}\frac{f}{g}(\sigma(x)) &= \sum_{j=1}^{J} A(h_i(s_j(x))) \\ &= \sum_{j=1}^{J} \prod_{h\in H} h(h_i(s_j(x))) \\ &= \sum_{j=1}^{J} \prod_{h\in H} h(s_j(x)) \\ &= \sum_{j=1}^{J} A(s_j(x)) = \frac{f}{g}(x).\end{aligned}$$

So, if $f(x_0) - t\,g(x_0) = 0$ for some t, x_0, then also $f(\sigma(x_0)) - t\,g(\sigma(x_0)) = 0$, for any $\sigma \in G$. This concludes our proof. □

For our purposes, we must fine-tune the selection of these polynomials. In particular, we want polynomials with integer coefficients, as given in the following corollary:

Corollary 6.2. *For $G \subset PGL(2,\mathbb{Q})$, G finite, there exist polynomials $f(x)$, $g(x)$ in $\mathbb{Z}[x]$ such that $C(f) = 1$, $\ell(f) \,|\, C(g)$, and f, g satisfy Theorem 6.1.*

Proof. Let f, g be as in Theorem 6.1, with K as $\mathbb{Q}$. So, $f, g \in \mathbb{Q}[x]$, and by multiplying f and g by appropriate constants, we can assume $f, g \in \mathbb{Z}[x]$ and $C(f) = C(g) = 1$. Multiply g by $\ell(f)$, the leading coefficient of f, and we are done. □

Our proof of Corollary 1.2 follows immediately:

Proof. Given G, let $f(x), g(x)$ be as described in Corollary 6.2. Then, for α a root of $A_t(x) = f(x) - t\,g(x)$, we use Theorem 6.1 and Corollary 6.2 to conclude that

$$\left\{\sum_{A_t(\alpha)=0} h(\alpha) : t \in \overline{\mathbb{Q}}\right\} \subset \left\{\sum_{\sigma\in G} h(\sigma(\alpha)) : \alpha \in \overline{\mathbb{Q}}\right\}.$$

By Theorem 1.1, the first set is dense in $[d, \infty)$ (for some d), and thus so is the second set. □

7. Examples

Let us first consider Zhang's and Zagier's set $\Lambda_2 = \{h(\alpha) + h(1-\alpha)\}$. Using $f(x) = x - x^2$ and $g(x) = 1$ (and $A_t(x) = f(x) - t\,g(x)$), we see that the set $\{\sum_{A_t(\alpha)=0} h(\alpha)\} \subset \Lambda_2$, as if α is a root of $A_t(x)$, so also is $1-\alpha$. Theorem 1.1 guarantees that $\{\sum_{A_t(\alpha)=0} h(\alpha)\}$ is dense in $[d, \infty)$, so we need only calculate d. To do so, we use $m(t)$ (from (3.5)) and the methods of section 5.2 to compute

$$d = \mathcal{L}_0 = \lim_{n\to\infty} \frac{1}{2^n} \sum_{i=1}^{2^n} \log m(\beta_{n,i}),$$

where the $\beta_{n,i}$ are as defined in section 4 (with $\beta_0 = b = 1$). For this example, $m(t)$ can be explicitly calculated as $m(t) = |t|$ for $t \leq -2$ or $t \geq 1$, $m(t) = 1$ for $0 \leq t < 1$, and $m(t) = \frac{1}{2} + \frac{\sqrt{1-4t}}{2}$ for $-2 < t < 0$. Using `Maple`, our d comes out to be about $0.39678\ldots$. Compare this to Zagier's discovery that the first nontrivial point in Λ_2 is $\frac{1}{2}\log\frac{1+\sqrt{5}}{2} = 0.2406\ldots$, and Doche's proof in [4] that Λ_2 is dense in $[0.2544\ldots,\infty]$.

Next, we consider the spectrum $\Lambda_3 = \{h(\alpha) + h(1-\frac{1}{\alpha}) + h(\frac{1}{1-\alpha})\}$, as first described in [6]. As with Λ_2 above, this set is discrete near zero; we showed in [6] that the first two values of Λ_3 are 0 (for $\alpha = 1$) and $0.4218\ldots$ (for α a root of $(x^2-x+1)^3 - (x^2-x)^2 = 0$.) The next two values are known to be $0.4336\ldots$ and $0.4380\ldots$, as seen in [5, Chapter 7]. Using $f(x) = x^3 - 3x + 1$ and $g(x) = x^2 - x$, we get a limit point (with density beyond that point) of about $0.55\ldots$. A better limit point of $0.44\ldots$ is known to exist, as seen in [5], and can be achieved by using $(\beta_{n,i})^2$ instead of $\beta_{n,i}$, and different polynomials for f and g in Section 4.

It's interesting to note that both of these examples could have been approached by way of Corollary 1.2. Zhang's and Zagier's spectrum Λ_2 is produced by the group generated by $\sigma(x) = 1 - x$. Our own Λ_3 comes from the group generated by $\sigma(x) = \frac{1}{1-x}$, and indeed we originally found the polynomials $f(x) = x^3 - 3x + 1$ and $g(x) = x^2 - x$ by using the methods given in the proof of Theorem 6.1.

We'd also like to mention that many of the above examples and theorems (in particular, Theorem 1.1) can be rewritten in terms of Mahler's measure. For example, if α has minimal polynomial $p(x)$, then, using the fact that M is multiplicative, we have $h(\alpha) + h(1-\alpha) = \frac{1}{\deg p}\log M(p(x)p(1-x))$. It's easy to show that $p(x)p(1-x) = R(x-x^2)$ for some polynomial R in $\mathbb{Z}[x]$ of the same degree as p, and so we have $h(\alpha) + h(1-\alpha) = \frac{1}{\deg R}\log M(R(x-x^2))$. This motivates us to turn to a recent article by G. Rhin and C. Smyth [9], which gives a (not necessarily sharp, but easily computed) positive lower bound for expressions of the form $\frac{1}{\deg R}\log M(R(T(x)))$, where $T(x)$ is a fixed polynomial of at least two terms and divisible by x, and where $R(x)$ ranges over all irreducible polynomials. This is easily extended to a lower bound for all polynomials $R(x)$, thus giving a nice extension of Zhang's result in [15].

As another application of Corollary 1.2, let us consider a family of groups, each of order six. Choose a pair of distinct rationals (a, b), let $c = a^2 - ab + b^2$, and construct a group G_6 generated by $\sigma(x) = (ax - c)/(x - b)$ and $\tau(x) = c/x$. It's not hard to show that σ and τ form a group of order six, and it's also easy to show that two polynomials that will satisfy the requirements of Theorem 6.1 are $f(x) = (x^2 - (a+b)x + c)^3$ and $g(x) = x(x-a)(x-b)(x-c/a)(x-c/b)$. Taking the particular case of $(a, b) = (3/2, 1/2)$, and using $4^3 f(x)$ and $4^3 2^3 6 g(x)$ as our polynomials, we get new f, g which will satisfy Corollary 6.2:

$$\begin{aligned} f(x) &= (4x^2 - 8x + 7)^3, \\ g(x) &= 64x(2x-3)(2x-1)(6x-7)(2x-7). \end{aligned}$$

We use `Maple` and equation (5.11) to get a limit point (with density beyond that) of $d = \log(64) + 4.9206\cdots \approx 9.08\ldots$. This is certainly not an optimal value! On the other hand, with $(a, b) = (1, 0)$, we get the order-three group that produces Λ_3 (discussed above) combined with $\tau(x) = 1/x$ in a semidirect product to give us a new group of order six. Since $h(x) = h(1/x)$, the spectrum of this new group is simply $2\Lambda_3$.

Finally, let us consider the polynomials $f(x) = x^4+2$ and $g(x) = x^2$. It's not hard to show that $f(x)/g(x) = f(-x)/g(-x) = f(\sqrt{2}/x)/g(\sqrt{2}/x)$, and thus our $A_t(x)$ (defined, as always, as $A_t(x) = f(x) - t\,g(x)$) has roots $\{\alpha, -\alpha, \sqrt{2}/\alpha, -\sqrt{2}/\alpha\}$. Using Maple again, we find that the set $\{h(\alpha) + h(-\alpha) + h(\sqrt{2}/\alpha) + h(-\sqrt{2}/\alpha)\}$ is dense beyond $d = 0.69908\ldots$, and since $h(x) = h(-x) = h(1/x)$ for all $x \in \overline{\mathbb{Q}}$, we have that $\{h(\alpha) + h(\alpha/\sqrt{2}) : \alpha \in \overline{\mathbb{Q}}\}$ is dense in $[0.34954\ldots, \infty)$. A moment's thought about the minimal polynomials of α and $\alpha/\sqrt{2}$ will convince us that the minimal positive value in this spectrum is $\frac{1}{2}\log 2 = 0.34657\ldots$. We thus conjecture that our density point of $0.34954\ldots$ is best possible.

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