Finite Subgroups of the Extended Modular Group

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Abstract

We show that in the extended modular group $\overline{\Gamma} = \mathrm{PGL}(2, \mathbb{Z})$ there are exactly seven finite subgroups up to conjugacy; three subgroups of size 2, one subgroup each of size 3, 4, and 6, and the trivial subgroup of size 1.

Key words: extended modular group, conjugacy class, finite subgroups.

1 Introduction.

We recall that *linear fractional transforms* (also known as *Möbius transforms*) are functions of the form

$$m(x) = \frac{ax+b}{cx+d}$$

such that $ad - bc \neq 0$. These functions form a group under composition, but we will be interested in the particular group of such functions with integer coefficients a, b, c, d and associated determinant equal to ± 1. We note that the map

$$\phi : \frac{ax+b}{cx+d} \quad \longmapsto \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is an isomorphism from this group of functions (with integer coefficients $ad - bc = \pm 1$) to the projective matrix group $\overline{\Gamma} = \mathrm{PGL}(2, \mathbb{Z})$, which is often referred to as the *extended modular group*. A particularly nice feature of this isomorphism is that composition of functions can be re-written in terms of multiplication of matrices. In what follows, we will use the function notation and the matrix notation interchangeably, with the further understanding that

$$\frac{ax+b}{cx+d} \qquad \text{is the same as} \qquad \frac{-ax-b}{-cx-d},$$

and likewise we understand that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{is the same as} \qquad \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}.$$

Although much has been written about this extended modular group (see, for example, [2, 3, 4, 5]), we are interested in the following question: what are the possible finite subgroups of $\overline{\Gamma} = \mathrm{PGL}(2,\mathbb{Z})$, up to conjugacy? The answer can be found by carefully utilizing two previous results from 2003 and 2004.

2 Statement of Main Result.

This first theorem, by Yılmaz Özgür and Şahin [6, Theorem 2.3], comes from considering the presentation of $\overline{\Gamma}$ as a free group with three generators, and it gives us *elements* in $\overline{\Gamma}$ of finite order, up to *conjugacy.*

Theorem 1 (Yılmaz Özgür, Şahin)**.** *There are exactly four conjugacy classes for non-trivial elements of finite order in $\overline{\Gamma}$. Every element of order two is conjugate to either $1/x$ or $-x$ or $-1/x$, and every element of order three is conjugate to $-1/(x+1)$.*

This second result, derived from a paper by Dresden [1], comes from considering the finite symmetry groups of the sphere, and it gives us *subgroups* in $\overline{\Gamma}$ of finite order, up to *isomorphism.*

Theorem 2. *There are exactly four isomorphism classes for non-trivial subgroups of finite order in $\overline{\Gamma}$. Every such subgroup is isomorphic to either one of the cyclic groups C_2, C_3, or one of the dihedral groups D_2, D_3, of sizes 2, 3, 4, and 6 respectively.*

(We will prove this theorem in a moment.) We will be able to combine these two theorems to prove our main result, which we state here.

Theorem 3. *Any finite non-trivial subgroup of $\overline{\Gamma} = \mathrm{PGL}(2,\mathbb{Z})$ is of size two, three, four, or six. The groups of size two are conjugate in $\overline{\Gamma}$ to either $\{x, -x\}$ or $\{x, 1/x\}$ or $\{x, -1/x\}$. All groups of size three in $\overline{\Gamma} = \mathrm{PGL}(2,\mathbb{Z})$ are conjugate in $\overline{\Gamma}$ to*

$$G_3 = \left\{x,\ \frac{-1}{x+1},\ \frac{-x-1}{x}\right\}.$$

Likewise, all groups of size four are conjugate to

$$G_4 = \left\{x,\ \frac{1}{x},\ -x,\ \frac{-1}{x}\right\}.$$

and all groups of size six are conjugate to

$$G_6 = \left\{x,\ \frac{-1}{x+1},\ \frac{-x-1}{x},\ \frac{1}{x},\ \frac{-x}{x+1},\ -x-1\right\}.$$

3 Proofs

Proof of Theorem 2. While $\overline{\Gamma} = \mathrm{PGL}(2,\mathbb{Z})$ consists of linear fractional transforms $(ax+b)/(cx+d)$ with integer coefficients and determinant $ad - bc = \pm 1$, we can also define the larger group $\mathrm{PGL}(2,\mathbb{Q})$ to be all such linear fractional transforms with integer coefficients but with determinant $ad - bc \neq 0$. By Theorem 1 of [1], we learn that all non-trivial finite subgroups of $\mathrm{PGL}(2,\mathbb{Q})$ are isomorphic to either C_2, C_3, C_4, C_6, D_2, D_3, D_4, or D_6. From Lemma 1 of the same paper, we learn that all finite elements of order 4 are conjugate (in $\mathrm{PGL}(2,\mathbb{Q})$) to $\dfrac{x-1}{x+1}$, and those of order 6 are conjugate (again, in $\mathrm{PGL}(2,\mathbb{Q})$) to $\dfrac{2x-1}{x+1}$. Since these have determinants 2 and 3 respectively, and since multiplication in $\mathrm{PGL}(2,\mathbb{Q})$ can change the determinant only up to a square factor, we can conclude that there can be no elements in $\mathrm{PGL}(2,\mathbb{Z})$ of order 4 or 6. This eliminates from consideration the groups C_4, C_6, D_4, and D_6.

It remains to show that the groups C_2, C_3, D_2, and D_3 are realizable in $\overline{\Gamma} = \mathrm{PGL}(2,\mathbb{Z})$. The cyclic group C_2 can be realized by $\{x, -x\}$, and C_3, D_2, and D_3 are represented by the groups G_3, G_4, and G_6 in Theorem 3. □

The following proposition is essential to our proof, and will allow us to stitch together Theorems 1 and 2.

Proposition 1. *If a subgroup of* $\overline{\Gamma} = \mathrm{PGL}(2,\mathbb{Z})$ *contains* $-x$ *and is of size 4, then it must equal* G_4. *Likewise, if a subgroup of* $\overline{\Gamma} = \mathrm{PGL}(2,\mathbb{Z})$ *contains* $-1/(x+1)$ *and is of size 6, then it must equal* G_6.

Proof. Suppose F_4 is a subgroup of size 4 in $\overline{\Gamma} = \mathrm{PGL}(2,\mathbb{Z})$ that contains $-x$ and some other function $p(x) = \dfrac{ax+b}{cx+d}$ different from x and $-x$. We know from Theorem 2 that F_4 is dihedral, which means

$$F_4 = \{x,\ p(x),\ -x,\ p(-x)\}.$$

In particular, $p(p(x))$ is required to equal x, and $p(-x)$ must equal $-p(x)$. From the first requirement we get the matrix equation

$$\begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & d^2+bc \end{pmatrix} = \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{1}$$

and from the second requirement we obtain the equation

$$\begin{pmatrix} -a & b \\ -c & d \end{pmatrix} = \pm\begin{pmatrix} -a & -b \\ c & d \end{pmatrix}. \tag{2}$$

We consider the two options from the above equation. If we have the "+" in the $\pm$ in equation (2), then $b = c = 0$, which means equation (1) gives us $a^2 = 1$ and $d^2 = 1$, and so $p(x)$ is either x or $-x$, a contradiction to out assumption. On the other hand, if we have the "$-$" in the $\pm$ in equation (2), then $a = d = 0$, which means equation (1) gives us $bc = \pm 1$, which implies $p(x)$ is either $1/x$ or $-1/x$. Either way, we get that F_4 is identical to G_4.

Now, suppose F_6 is a subgroup of size 6 in $\overline{\Gamma} = \mathrm{PGL}(2,\mathbb{Z})$ that contains $m(x) = -1/(x+1)$. From Theorem 2 we know that F_6 is dihedral, and hence it must also contain some function $p(x) = \dfrac{ax+b}{cx+d}$ of order two, such that

$$F_6 = \{x,\ m(x),\ m(m(x)),\ p(x),\ p(m(x)),\ m(p(x))\}.$$

In particular, $p(p(x))$ is required to equal x, and $p(m(x))$ must equal $m(m(p(x))$. The first requirement gives us equation (1), and from the second requirement we obtain the equation

$$\begin{pmatrix} b & -a+b \\ d & -c+d \end{pmatrix} = \pm\begin{pmatrix} -a-c & -b-d \\ a & b \end{pmatrix}. \tag{3}$$

If we have the "+" in the $\pm$ in equation (3), we quickly solve to find $a = b = c = d = 0$, a contradiction to $ad - bc = \pm 1$. On the other hand, if we have the "$-$" in the $\pm$ in equation (3), then we get $b = a + c$ and $d = -a$, which when substituted into $ad - bc = \pm 1$ becomes $a^2 - ac + c^2 = \mp 1$, and this only has a solution for $a^2 - ac + c^2 = 1$; we also get from equation (1) that $a^2 + ac + c^2 = 1$, and combining these two equations and enumerating the possible values for a and c leads us to conclude that $p(x)$ is either $-x - 1$, or $1/x$, or $-x/(x+1)$. In all cases, we get that F_6 is identical to G_6. □

It is now an easy matter to prove our main result.

Proof of Theorem 3. The case for groups of size two and three follows immediately from Theorem 1. If G is a group of size four, then by Theorem 2 it is dihedral with three elements of order two; by Theorem 1 one of those elements is conjugate to $-x$ and so we can conjugate our group G to get a new group G' containing $-x$; by Proposition 1 this new group G' must equal G_4. Finally, if G is a group of size 6, then by Theorem 2 it is dihedral with an element of order three; by Theorem 1 we can conjugate it to get a new group G' containing $-1/(x+1)$, and by Proposition 1 this new group G' must equal G_6. □

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