Cubic Polynomials, Linear Shifts, and Ramanujan Cubics.

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Abstract

We show that every monic polynomial of degree three with complex coefficients and no repeated roots is either a (vertical and horizontal) translation of $y = x^3$ or can be composed with a linear function to obtain a Ramanujan cubic. As a result, we gain some new insights into the roots of cubic polynomials.

1 Introduction.

It's hard to pick out a favorite from Ramanujan's nearly-uncountable collection of delightful identities, but these two have to be near the top of anyone's list:

$$\sqrt[3]{1/9} \;-\; \sqrt[3]{2/9} \;+\; \sqrt[3]{4/9} \;=\; \sqrt[3]{\sqrt[3]{2}-1} \tag{1}$$

$$\sqrt[3]{\cos\frac{2\pi}{9}} \;+\; \sqrt[3]{\cos\frac{4\pi}{9}} \;-\; \sqrt[3]{\cos\frac{\pi}{9}} \;=\; \sqrt[3]{\frac{3}{2}\left(\sqrt[3]{9}-2\right)} \tag{2}$$

Both of these equations appear in Ramanujan's notebooks [2], and they have been studied in a number of papers. Landau [8, 9] treated the first equation as an example of how a "nested radical" like $\sqrt[3]{\sqrt[3]{2}-1}$ can be "de-nested" into a sum of simple cube roots. Berndt and Bhargava [3] gave a proof of the second equation using only elementary methods. Shevelev [15] provided a delightful and elementary proof of both formulas (and quite a few others). It turns out that both (1) and (2) are related to the roots of a special class of degree-3 polynomials, and in keeping with past work [1, 16, 17], we will define a *Ramanujan simple cubic* (RSC) to be a polynomial with (possibly complex) coefficients of the form

$$p_B(x) = x^3 - \left(\tfrac{3+B}{2}\right)x^2 - \left(\tfrac{3-B}{2}\right)x + 1.$$

We will use a technique from the mid-1800's to prove that almost every cubic is just a linear shift away from a Ramanujan simple cubic, and this will allow us to recapture the two formulas above, and also to come up with some lovely new identities, such as the deceptively simple formula

$$2\sqrt{6}\cos\frac{11\pi}{36} \;+\; 6\cos\frac{10\pi}{36} \;=\; \left(3\sqrt{2}+\sqrt{6}\right)\cos\frac{\pi}{36} \tag{3}$$

and the rather surprising fact that

$$\frac{1}{2}\left(-5+\sqrt{13}+2\sqrt{26-6\sqrt{13}}\cos\frac{\pi}{26}\right) \tag{4}$$

is a solution to $x^3 + x^2 - 4x + 1 = 0$

In section 2 we discuss the properties of these RSC polynomials, and in section 3 we prove our main result. This will lead to many nice examples in section 4.

2 Ramanujan Simple Cubics.

Surprisingly, these RSC's have been studied in one form or another for over a hundred years. In 1911, Dickson [4] discussed integral solutions to $p_B(x) = 0$ modulo a prime. More recently, a number of authors [1, 16, 17] have studied a slightly more general class of polynomials they call *Ramanujan cubics*, which are simply our RSC polynomials $p_B(x)$ but with x replaced by x/s. Similarly, if we replace the x in $p_B(x)$ with $-x$, we get the *Shanks polynomials*, so called because they generate what Shanks called the "simplest cubic fields" [14]. Foster's paper [6] has an excellent review of earlier work on the Shanks polynomials and the simplest cubic fields; he also proved that every degree-three cyclic extension of the rationals is generated by a Shanks polynomial (which implies the same for our RSC); this was done earlier by Kersten and Michaliček [7]. Also, Lehmer [11] and Lazarus [10] have shown that the minimal polynomials for so-called *cubic Gaussian periods*, when composed with some $x - a$ for a an integer, will equal one of the Shanks polynomials (and thus are related to our RSC's).

The following theorem illustrates some of the remarkable properties of Ramanujan simple cubics (RSC).

Theorem 1. *For $p_B(x) = x^3 - \left(\frac{3+B}{2}\right)x^2 - \left(\frac{3-B}{2}\right)x + 1$ the Ramanujan simple cubic defined earlier,*

1. *The roots r_1, r_2, r_3 of $p_B(x)$ are always permuted by the order-three map $n(x) = \frac{1}{1-x}$.*
2. *The roots r_1, r_2, r_3 satisfy*

$$\sqrt[3]{r_1} \; + \; \sqrt[3]{r_2} \; + \; \sqrt[3]{r_3} \; = \; \sqrt[3]{\left(\frac{3+B}{2}\right) \; - \; 6 \; + \; 3\sqrt[3]{\frac{27+B^2}{4}}} \tag{5}$$

   so long as, for complex arguments, we choose the appropriate values for the cube roots.

3. *If we define the elements of the set $\{s_1, s_2, \ldots, s_6\}$ as*

$$s_k \; = \; \frac{1}{3}\left(\left(\frac{3+B}{2}\right) \; + \; \sqrt{27+B^2}\; \cos\left(\frac{k\pi}{3} + \frac{1}{3}\arctan\frac{3\sqrt{3}}{B}\right)\right) \tag{6}$$

   then for $B \geq 0$ the roots of $p_B(x)$ are $\{s_2, s_4, s_6\}$ and for $B \leq 0$ the roots of $p_B(x)$ are $\{s_1, s_3, s_5\}$.

Remark. A more complicated version of equation (6) for the roots of a general cubic has been known since the times of Viète and Descartes; our presentation serves to illustrate how much simpler this formulation can be when dealing with the Ramanujan simple cubics. Also, while (6) is not actually defined at $B = 0$, we can interpret it at that value by simply taking the limit of (6) as B approaches 0. Surprisingly, whether we have B approach 0 from above or from below, the three values of $\{s_2, s_4, s_6\}$ and the three values of $\{s_1, s_3, s_5\}$ coincide at $\{-1, 1/2, 2\}$, which are indeed the three roots of $p_0(x) = x^3 - 3/2\,x^2 - 3/2\,x + 1$.

Proof. For part 1, a quick calculation gives us that $-p_B\left(\frac{1}{1-x}\right) \cdot (1-x)^3 = p_B(x)$. Since 1 is never a root of $p_B(x)$, this shows that if r_1 is a root of p_B then so also is $\frac{1}{1-r_1}$. This is enough to show that $n(x)$ permutes the roots so long as $\frac{1}{1-r_1}$ is different from r_1; in the case where $\frac{1}{1-r_1}$ equals r_1, this implies that r_1 is a primitive sixth root of unity, which means $B = \pm i\sqrt{27}$ and all three roots of $p_B(x)$ are identical (and hence, technically, are still "permuted" by $\frac{1}{1-x}$).

For part 2, there is an elementary proof in [3, p. 652] of a nearly identical statement for the roots of the Shanks polynomial $x^3 - ax^2 - (a+3)x - 1$; the roots of this Shanks polynomial are the negatives of the roots of the Ramanujan polynomial $p_B(x) = x^3 + ax^2 - (a+3)x + 1$ with $B = -2a - 3$ and so the identity follows. This proof also appears in [2, p. 22].

For part 3, we refer the reader to the similar proof for Shanks polynomials in [1, Theorem 7]; another version of this formula (without proof, and for just one root) can be found in [12]. □

Example 1. We can now easily show that equations (1) and (2) arise from equation (5) of Theorem 1. For equation (1), we take $p_B(x)$ with $B = 0$ which has roots $1/2, -1, 2$, and so equation (5) gives us

$$\sqrt[3]{1/2} \ + \ \sqrt[3]{-1} \ + \ \sqrt[3]{2} \ = \ \sqrt[3]{\left(\tfrac{3}{2}\right) \ - \ 6 \ + \ 3\sqrt[3]{\tfrac{27}{4}}}$$

and after multiplying through by $\sqrt[3]{2/9}$ and doing some simplifying on the right, we get the desired equation.

As for (2), we note that the minimal polynomial for $2\cos 2\pi/9$ is $x^3 - 3x + 1$, a Ramanujan simple cubic with $B = -3$. It's easy to show that the other two roots are $2\cos 4\pi/9$ and $-2\cos \pi/9$, and so equation (5) gives us

$$\sqrt[3]{2\cos 2\pi/9} \ + \ \sqrt[3]{2\cos 4\pi/9} \ + \ \sqrt[3]{-2\cos \pi/9} \ = \ \sqrt[3]{\left(\tfrac{3-3}{2}\right) \ - \ 6 \ + \ 3\sqrt[3]{\tfrac{27+9}{4}}}$$

and after simplifying the right and dividing by $\sqrt[3]{2}$ we obtain the desired formula.

In the previous example, we began with a particular Ramanujan simple cubic and then derived statements about its roots. We can reverse the process, as seen next.

Example 2. Suppose we wish to create a Ramanujan simple cubic with $x_1 = \sqrt{3} - 1$ as one of its roots. We know that the other two roots must satisfy $x_2 = n(x_1)$ and $x_3 = n(x_2)$, where $n(x) = \frac{1}{1-x}$. This leads to $x_2 = 2 + \sqrt{3}$ and $x_3 = (1 - \sqrt{3})/2$, and the polynomial $(x - x_1)(x - x_2)(x - x_3)$ is easily calculated to be a Ramanujan simple cubic with $B = 3\sqrt{3}$. This leads to a particularly nice formulation of equation (5); after some simplification (and after multiplying through by $\sqrt[3]{2}$ on both sides) we obtain the following unexpected equation:

$$\sqrt[3]{2\sqrt{3} - 2} \ - \ \sqrt[3]{\sqrt{3} - 1} \ + \ \sqrt[3]{2\sqrt{3} + 4} \ = \ \sqrt{3} \cdot \sqrt[3]{1 + \sqrt{3}\left(\sqrt[3]{4} - 1\right)}.$$

3 Main Result.

For $f(x) = x^3 + Px^2 + Qx + R$ a polynomial with (possibly) complex coefficients, we note that its discriminant is

$$\Delta = P^2Q^2 - 4Q^3 - 4P^3R + 18PQR - 27R^2,$$

and we recall that a polynomial has no repeated roots if and only if its discriminant Δ is not zero. With this in mind, we define the following two values (taken from their original definitions in [13, p. 468]):

$$a = \frac{\sqrt{\Delta} - (9R - PQ)}{2\sqrt{\Delta}},$$

$$c = \frac{6Q - 2P^2}{2\sqrt{\Delta}}.$$

We now state our main result.

Theorem 2. *Let $f(x) = x^3 + Px^2 + Qx + R$ have non-repeated roots t_1, t_2, t_3, and let a and c be as defined above.*

1. *If $c = 0$, then there exists h and k such that $f(x) = (x-h)^3 + k$. In other words, $f(x)$ is a translation of x^3 (by h units horizontally and k units vertically).*

2. *If $c \neq 0$, then $f\left(\frac{a-x}{c}\right) \cdot (-c)^3$ equals the Ramanujan simple cubic $p_B(x) = x^3 - \left(\frac{3+B}{2}\right)x^2 - \left(\frac{3-B}{2}\right)x + 1$, with $B = 6a + 2cP - 3$. In particular, the set of roots of $p_B(x)$ are $\{a - c\cdot t_1,\ a - c\cdot t_2,\ a - c\cdot t_3\}$.*

Proof. First, suppose $c = 0$. Then $Q = P^2/3$ and so $f(x)$ can be written as $(x-h)^3 + k$ with $h = -P/3$ and $k = R - P^3/3$.

Next, suppose $c \neq 0$. It is possible to use brute force to show that $f\left(\frac{a-x}{c}\right) \cdot (-c)^3$ equals $p_B(x)$, but that does not provide much insight into the problem. Instead, we offer the following more detailed explanation. The key can be found in Serret's classic algebra textbook [13, p. 468] from the mid nineteenth century. In pursuit of an entirely unrelated problem, Serret defined the a and c seen above, along with the following:

$$b = \frac{2Q^2 - 6PR}{2\sqrt{\Delta}}$$

$$d = 1 - a$$

Serret showed that $m(x) = \frac{ax+b}{cx+d}$ is of order three under composition, permutes the roots t_1, t_2, t_3 of the cubic $f(x) = x^3 + Px^2 + Qx + R$, and has the property that $ad - bc = 1$. Now, we would like to transform the cubic $f(x)$ into a new cubic whose roots are permuted by $n(x) = \frac{1}{1-x}$, and one way to do that is to first find a linear map $q(x)$ such that $(q^{-1} \circ m \circ q)(x) = n(x)$, and then to consider the composition $(f \circ q)(x)$. This composition would have as roots the numbers $q^{-1}(t_1), q^{-1}(t_2), q^{-1}(t_3)$, and furthermore these roots would be permuted by $(q^{-1} \circ m \circ q)(x) = \frac{1}{1-x}$. We can then show this composition must be a Ramanujan simple cubic.

With this in mind, it remains to find our $q(x)$ such that $(q^{-1} \circ m \circ q)(x) = n(x)$. This is a fairly easy task if one uses the language of Möbius transforms (see, for example, [5]). Since $n(x)$ takes ∞ to 0 to 1 back to ∞, and $m(x)$ takes ∞ to a/c to $-d/c$ back to ∞, we can choose $q(x)$ to take ∞ to ∞, and 0 to a/c, and 1 to $-d/c$. This gives us $q(x) = \frac{a-x}{c}$ and $q^{-1}(x) = a - cx$. We can verify that indeed $(q^{-1} \circ m \circ q)(x) = n(x)$, and that $f(q(x)) = f\left(\frac{a-x}{c}\right)$ has roots $a - ct_1$, $a - ct_2$, and $a - ct_3$ as desired. Since these are permuted by $\frac{1}{1-x}$, it's easy to show that the monic polynomial $f(q(x)) \cdot (-c)^3$ has the appropriate coefficients to match the desired form of a Ramanujan simple cubic. □

We can now combine Theorem 2 with Theorem 1 to give us the following results.

Corollary 1. *Let $f(x) = x^3 + Px^2 + Qx + R$ have non-repeated roots t_1, t_2, t_3, and let a, B, and c be as defined in Theorem 2, with $c \neq 0$. Then,*

1. *The order-three map $n(x) = \frac{1}{1-x}$ permutes the set $\{a - c\cdot t_1,\ a - c\cdot t_2,\ a - c\cdot t_3\}$.*

2. *We have the Ramanujan-style equation*

$$\sqrt[3]{a - c\cdot t_1} + \sqrt[3]{a - c\cdot t_2} + \sqrt[3]{a - c\cdot t_3} = \sqrt[3]{\left(\frac{3+B}{2}\right) - 6 + 3\sqrt[3]{\frac{27+B^2}{4}}}, \tag{7}$$

so long as, for complex arguments, we choose the appropriate values for the cube roots.

3. *If we define the elements of the set $\{u_1, u_2, \ldots, u_6\}$ as*

$$u_k = \frac{a}{c} - \frac{1}{3c}\left(\left(\frac{3+B}{2}\right) + \sqrt{27+B^2}\, \cos\left(\frac{k\pi}{3} + \frac{1}{3}\arctan\frac{3\sqrt{3}}{B}\right)\right) \tag{8}$$

then for $B \geq 0$ the roots of $f(x)$ are $\{u_2, u_4, u_6\}$ and for $B \leq 0$ the roots of $f(x)$ are $\{u_1, u_3, u_5\}$.

We note that a similar version of formula (7) was presented (without proof) by forum user Tito Piezas III on `math.stackexchange.com`.

4 Examples.

Example 3. Here's a rather lovely formula which we believe has not been seen before:

$$\sqrt[3]{3-\sqrt{21}+8\cos\frac{2\pi}{21}} \;+\; \sqrt[3]{3-\sqrt{21}+8\cos\frac{8\pi}{21}} \;+\; \sqrt[3]{3-\sqrt{21}+8\cos\frac{10\pi}{21}} = \sqrt[3]{-1-\sqrt{21}+6\sqrt[3]{28-4\sqrt{21}}}.$$

To obtain this, we begin with $x^6-x^5-6x^4+6x^3+8x^2-8x+1$, the minimal polynomial for $t_1 = 2\cos 2\pi/21$. This factors in $\mathbb{Q}(\sqrt{21})$ as two cubics, and we choose the one which still has $2\cos 2\pi/21$ as a root. This cubic is $x^3+\frac{1}{2}\left(-1-\sqrt{21}\right)x^2+\frac{1}{2}\left(\sqrt{21}-1\right)x+\frac{1}{2}\left(\sqrt{21}-5\right)$, and its other two roots are $t_2 = 2\cos 8\pi/21$ and $t_3 = 2\cos 10\pi/21$, and after doing the computations in Theorem 2 we obtain $a = \frac{1}{2}\left(3-\sqrt{21}\right)$, $c=-2$, and $B = 8-\sqrt{21}$. We then plug these values into formula (7), multiply through by $\sqrt[3]{2}$, and apply a few simplifications to obtain the above expression.

Example 4. We can do similar calculations for $2\cos\pi/18$. This has a minimal polynomial of degree 6, but it factors in $\mathbb{Q}(\sqrt{3})[x]$ and we choose the degree-three factor $g(x) = x^3-3x-\sqrt{3}$. One root of $g(x)$ is indeed $2\cos\pi/18$, and the other two roots are $2\cos 11\pi/18$ and $2\cos 13\pi/18$. Calculating a, c as defined in Section 3, we get $a=2$ and $c=-\sqrt{3}$. Thus, by Theorem 2, we have $g(\frac{a-x}{c})\cdot(-c)^3 = x^3-6x^2+3x+1$ which is a particularly nice Ramanujan simple cubic with $B=9$. (We will return to this cubic in Example 6.) By Corollary 1, we get a nice identity:

$$\sqrt[3]{2+2\sqrt{3}\cos\frac{\pi}{18}}+\sqrt[3]{2+2\sqrt{3}\cos\frac{11\pi}{18}}+\sqrt[3]{2+2\sqrt{3}\cos\frac{13\pi}{18}} = \sqrt[3]{9}. \tag{9}$$

Furthermore, by Theorem 1, we know the roots of x^3-6x^2+3x+1 are permuted by $1/(1-x)$. Therefore, by choosing our roots carefully, we get

$$2+2\sqrt{3}\cos\frac{13\pi}{18} = \frac{1}{1-\left(2+2\sqrt{3}\cos\dfrac{\pi}{18}\right)}$$

and this simplifies to

$$2\cos\frac{\pi}{18}+\cos\frac{13\pi}{18}+\sqrt{3}\cos\frac{14\pi}{18} = 0$$

which reduces to

$$\cos\frac{5\pi}{18} \;=\; 2\cos\frac{\pi}{18}-\sqrt{3}\cos\frac{4\pi}{18}. \tag{10}$$

Example 5. In an effort to find more equations like (10), we look at the minimal polynomials for $2\cos\pi/36$ and $2\cos\pi/42$. Both have minimal polynomials of degree 12, and both can be factored down into degree three polynomials by adjoining appropriate square roots to the rationals. By following the same steps as in the previous example we can arrive at the following two identities:

$$2\sqrt{6}\cos\frac{11\pi}{36} \;+\; 6\cos\frac{10\pi}{36} \;-\; \left(3\sqrt{2}+\sqrt{6}\right)\cos\frac{\pi}{36} = 0 \tag{11}$$

$$(\sqrt{3}-\sqrt{7})\cos\frac{\pi}{42} \;-\; 2\sqrt{7}\cos\frac{25\pi}{42} \;-\; 8\cos\frac{\pi}{42}\cos\frac{25\pi}{42} \;=\; 3. \tag{12}$$

It's probably just a coincidence, but $\left(3\sqrt{2}+\sqrt{6}\right)\cos\pi/36$ from formula (11) is almost identical (to six decimal places) to $20/3$. Also, note that (11) is equation (3) from the beginning of the article.

Example 6. Returning our attention to Example 4, we note that Theorem 2 gave us the particularly nice cubic $x^3 - 6x^2 + 3x + 1$ and gave us that one of its roots is $2 + 2\sqrt{3}\cos\pi/18$. Likewise, if we begin with $2\cos\pi/26$, we can factor its minimal (degree-12) polynomial down to a degree-3 polynomial with irrational coefficients, apply Theorem 2, and end up with another particularly nice polynomial, this time x^3+x^2-4x+1, one of whose roots is given in equation (4) at the beginning of this paper.

It turns out that, as seen in [11], these two polynomials are also just an integer shift from the minimal polynomials for certain *cubic Gaussian periods*. The exact nature of these objects is beyond the scope of this article; for our purposes, we can consider them to be sums of roots of unity with their inverses. Suffice it to say that this recognition leads us to discover that $x^3 - 6x^2 + 3x + 1$ is the minimal polynomial for the following three numbers:

$$\left\{2 + 2\cos\frac{\pi}{9} + 2\cos\frac{2\pi}{9}, \quad 2 + 2\cos\frac{4\pi}{9} + 2\cos\frac{7\pi}{9}, \quad 2 + 2\cos\frac{5\pi}{9} + 2\cos\frac{8\pi}{9}\right\}.$$

Comparing these with $2 + 2\sqrt{3}\cos\pi/18$ leads us to the identity

$$2 + 2\sqrt{3}\cos\frac{\pi}{18} \;=\; 2 + 2\cos\frac{\pi}{9} + 2\cos\frac{2\pi}{9}.$$

Unfortunately, this simplifies to a triviality. However, along these lines, we also discover that x^3+x^2-4x+1 is the minimal polynomial for the following three numbers:

$$\left\{2\cos\frac{2\pi}{13} + 2\cos\frac{10\pi}{13}, \quad 2\cos\frac{4\pi}{13} + 2\cos\frac{6\pi}{13}, \quad 2\cos\frac{8\pi}{13} + 2\cos\frac{12\pi}{13}\right\}.$$

After comparing to the solution in equation (4), we obtain this (non-trivial) identity,

$$-5 \;+\; \sqrt{13} \;+\; 2\sqrt{26 - 6\sqrt{13}}\,\cos\frac{\pi}{26} \;=\; 4\cos\frac{4\pi}{13} \;+\; 4\cos\frac{6\pi}{13},$$

and this really is a lovely formula.

Example 7. We finish with an example that does not involve cosines. Consider the polynomial $f(x) = (x-1)(x-\sqrt{2})(x+\sqrt{3})$. This is not Ramanujan, but when we apply the methods of Theorem 2 we obtain a Ramanujan polynomial $p_B(x)$ with $B = -6 - \sqrt{2} - 5\sqrt{3} + \sqrt{6}$, and one of its roots is $-1 - \sqrt{2} - \sqrt{3} - \sqrt{6}$. After trying various values of k with formula (6), we find that

$$-1 - \sqrt{2} - \sqrt{3} - \sqrt{6} = \frac{1}{3}\left(\left(\frac{3+B}{2}\right) \;+\; \sqrt{27+B^2}\;\cos\left(-\pi + \frac{1}{3}\arctan\frac{3\sqrt{3}}{B}\right)\right)$$

and after applying our value of B and simplifying, we obtain the following formula:

$$3 + 5\sqrt{2} + \sqrt{3} + 7\sqrt{6} = 2\sqrt{2\left(73 - 9\sqrt{2} + 28\sqrt{3} - \sqrt{6}\right)}\cos\left(\frac{1}{3}\tan^{-1}\left(\frac{3\sqrt{3}}{-6-\sqrt{2}-5\sqrt{3}+\sqrt{6}}\right)\right),$$

and this is surprising if for no other reason than the relatively small size of the coefficients.

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