

# A Simplified Binet Formula for $k$ -Generalized Fibonacci Numbers

Gregory P. B. Dresden  
Department of Mathematics  
Washington and Lee University  
Lexington, VA 24450  
[dresdeng@wlu.edu](mailto:dresdeng@wlu.edu)

Zhaohui Du  
Shanghai, China  
[zhao.hui.du@gmail.com](mailto:zhao.hui.du@gmail.com)

## Abstract

In this paper, we present a Binet-style formula that can be used to produce the  $k$ -generalized Fibonacci numbers (that is, the Tribonaccis, Tetranaccis, etc.). Furthermore, we show that in fact one needs only take the integer closest to the first term of this Binet-style formula in order to generate the desired sequence.

## 1 Introduction

Let  $k \geq 2$  and define  $F_n^{(k)}$ , the  $n^{\text{th}}$   $k$ -generalized Fibonacci number, as follows:

$$F_n^{(k)} = \begin{cases} 0, & \text{if } n < 1; \\ 1, & \text{if } n = 1; \\ F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)}, & \text{if } n > 1 \end{cases}$$

These numbers are also called generalized Fibonacci numbers of order  $k$ , Fibonacci  $k$ -step numbers, Fibonacci  $k$ -sequences, or  $k$ -bonacci numbers. Note that for  $k = 2$ , we have  $F_n^{(2)} = F_n$ , our familiar Fibonacci numbers. For  $k = 3$  we have the so-called Tribonaccis (sequence number [A000073](#) in Sloane's *Encyclopedia of Integer Sequences*), followed by the Tetranaccis ([A000078](#)) for  $k = 4$ , and so on. According to Kessler and Schiff [6], these numbers also appear in probability theory and in certain sorting algorithms. We present here a chart of these numbers for the first few values of  $k$ :

$k$	name	first few non-zero terms
2	Fibonacci	1, 1, 2, 3, 5, 8, 13, 21, 34, ...
3	Tribonacci	1, 1, 2, 4, 7, 13, 24, 44, 81, ...
4	Tetranacci	1, 1, 2, 4, 8, 15, 29, 56, 108, ...
5	Pentanacci	1, 1, 2, 4, 8, 16, 31, 61, 120, ...

We remind the reader of the famous Binet formula (also known as the de Moivre formula) that can be used to calculate  $F_n$ , the Fibonacci numbers:

$$\begin{aligned} F_n &= \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right] \\ &= \frac{\alpha^n - \beta^n}{\alpha - \beta} \end{aligned}$$

for  $\alpha > \beta$  the two roots of  $x^2 - x - 1 = 0$ . For our purposes, it is convenient (and not particularly difficult) to rewrite this formula as follows:

$$F_n = \frac{\alpha - 1}{2 + 3(\alpha - 2)} \alpha^{n-1} + \frac{\beta - 1}{2 + 3(\beta - 2)} \beta^{n-1} \quad (1)$$

We leave the details to the reader.

Our first (and very minor) result is the following representation of  $F_n^{(k)}$ :

**Theorem 1.** For  $F_n^{(k)}$  the  $n^{\text{th}}$   $k$ -generalized Fibonacci number, then

$$F_n^{(k)} = \sum_{i=1}^k \frac{\alpha_i - 1}{2 + (k + 1)(\alpha_i - 2)} \alpha_i^{n-1} \quad (2)$$

for  $\alpha_1, \dots, \alpha_k$  the roots of  $x^k - x^{k-1} - \dots - 1 = 0$ .

This is a new presentation, but hardly a new result. There are many other ways of representing these  $k$ -generalized Fibonacci numbers, as seen in the articles [2, 3, 4, 5, 7, 8, 9]. Our Eq. (2) of Theorem 1 is perhaps slightly easier to understand, and it also allows us to do some analysis (as seen below). We point out that for  $k = 2$ , Eq. (2) reduces to the variant of the Binet formula (for the standard Fibonacci numbers) from Eq. (1).

As shown in three distinct proofs [9, 10, 13], the equation  $x^k - x^{k-1} - \dots - 1 = 0$  from Theorem 1 has just one root  $\alpha$  such that  $|\alpha| > 1$ , and the other roots are strictly inside the unit circle. We can conclude that the contribution of the other roots in Eq. 2 will quickly become trivial, and thus:

$$F_n^{(k)} \approx \frac{\alpha - 1}{2 + (k + 1)(\alpha - 2)} \alpha^{n-1} \quad \text{for } n \text{ sufficiently large.} \quad (3)$$

It's well known that for the Fibonacci sequence  $F_n^{(2)} = F_n$ , the "sufficiently large"  $n$  in Eq. (3) is  $n = 0$ , as shown here:

$n$	0	1	2	3	4	5	6
$F_n$	0	1	1	2	3	5	8
$\frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n$	0.447	0.724	1.171	1.894	3.065	4.960	8.025
error	.447	.277	.171	.106	.065	.040	.025

It is perhaps surprising to discover that a similar statement holds for all the  $k$ -generalized Fibonacci numbers. Let's first define  $\text{rnd}(x)$  to be the value of  $x$  rounded to the nearest integer:  $\text{rnd}(x) = \lfloor x + \frac{1}{2} \rfloor$ . Then, our main result is the following:

**Theorem 2.** For  $F_n^{(k)}$  the  $n^{\text{th}}$   $k$ -generalized Fibonacci number, then

$$F_n^{(k)} = \text{rnd} \left( \frac{\alpha - 1}{2 + (k + 1)(\alpha - 2)} \alpha^{n-1} \right)$$

for all  $n \geq 2 - k$  and for  $\alpha$  the unique positive root of  $x^k - x^{k-1} - \dots - 1 = 0$ .

We point out that this theorem is not as trivial as one might think. Note the error term for the generalized Fibonacci numbers of order  $k = 6$ , as seen in the following chart; it is not monotone decreasing in absolute value.

$n$	0	1	2	3	4	5	6	7
$F_n^{(6)}$	0	1	1	2	4	8	16	32
$\frac{\alpha-1}{2+7(\alpha-2)}\alpha^5$	0.263	0.522	1.035	2.053	4.072	8.078	16.023	31.782
error	.263	.478	.035	.053	.072	.078	.023	.218

We also point out that not every recurrence sequence admits such a simple formula as seen in Theorem 2. Consider, for example, the scaled Fibonacci sequence 10, 10, 20, 30, 50, 80, ..., which has Binet formula:

$$\frac{10}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{10}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n.$$

This can be written as  $\text{rnd} \left( \frac{10}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n \right)$ , but only for  $n \geq 5$ . As another example, the sequence 1, 2, 8, 24, 80, ... (defined by  $G_n = 2G_{n-1} + 4G_{n-2}$ ) can be written as

$$G_n = \frac{(1 + \sqrt{5})^n}{2\sqrt{5}} - \frac{(1 - \sqrt{5})^n}{2\sqrt{5}},$$

but because both  $1 + \sqrt{5}$  and  $1 - \sqrt{5}$  have absolute value greater than 1, then it would be impossible to express  $G_n$  in terms of just one of these two numbers.

## 2 Previous Results

We point out that for  $k = 3$  (the Tribonacci numbers), our Theorem 2 was found earlier by Spickerman [11]. His formula (modified slightly to match our notation) reads as follows, where  $\alpha$  is the real root, and  $\sigma$  and  $\bar{\sigma}$  are the two complex roots, of  $x^3 - x^2 - x - 1 = 0$ :

$$F_n^{(3)} = \text{rnd} \left( \frac{\alpha^2}{(\alpha - \sigma)(\alpha - \bar{\sigma})} \alpha^{n-1} \right) \tag{4}$$

It is not hard to show that for  $k = 3$ , our coefficient  $\frac{\alpha-1}{2+(k+1)(\alpha-2)}$  from Theorem 2 is equal to Spickerman's coefficient  $\frac{\alpha^2}{(\alpha-\sigma)(\alpha-\bar{\sigma})}$ . We leave the details to the reader.

In a subsequent article [12], Spickerman and Joyner developed a more complex version of our Theorem 1 to represent the generalized Fibonacci numbers. Using our notation, and with  $\{\alpha_i\}$  the set of roots of  $x^k - x^{k-1} - \dots - 1 = 0$ , their formula reads

$$F_n^{(k)} = \sum_{i=1}^k \frac{\alpha_i^{k+1} - \alpha_i^k}{2\alpha_i^k - (k+1)} \alpha_i^{n-1} \quad (5)$$

It is surprising that even after calculating out the appropriate constants in their Eq. (5) for  $2 \leq k \leq 10$ , neither Spickerman nor Joyner noted that they could have simply taken the first term in Eq. (5) for all  $n \geq 0$ , as Spickerman did in Eq. (4) for  $k = 3$ .

The Spickerman-Joyner Eq. (5) was extended by Wolfram [13] to the case with arbitrary starting conditions (rather than the initial sequence  $0, 0, \dots, 0, 1$ ). In the next section we will show that our Eq. (2) in Theorem 1 is equivalent to the Spickerman-Joyner formula given above (and thus is a special case of Wolfram's formula).

Finally, we note that the polynomials  $x^k - x^{k-1} - \dots - 1$  in Theorem 1 have been studied rather extensively. They are irreducible polynomials with just one zero outside the unit circle. That single zero is located between  $2(1 - 2^{-k})$  and  $2$  (as seen in Wolfram's article [13]; Miles [9] gave earlier and less precise results). It is also known [13, Lemma 3.11] that the polynomials have Galois group  $S_k$  for  $k \leq 11$ ; in particular, their zeros can not be expressed in radicals for  $5 \leq k \leq 11$ . Wolfram conjectured that the Galois group is always  $S_k$ . Cipu and Luca [1] were able to show that the Galois group is not contained in the alternating group  $A_k$ , and for  $k \geq 3$  it is not 2-nilpotent. They point out that this means the zeros of the polynomials  $x^k - x^{k-1} - \dots - 1$  for  $k \geq 3$  can not be constructed by ruler and compass, but the question of whether they are expressible using radicals remains open for  $k \geq 12$ .

### 3 Preliminary Lemmas

First, a few statements about the the number  $\alpha$ .

**Lemma 3.** *Let  $\alpha > 1$  be the real positive root of  $x^k - x^{k-1} - \dots - x - 1 = 0$ . Then,*

$$2 - \frac{1}{k} < \alpha < 2 \quad (6)$$

*In addition,*

$$2 - \frac{1}{3k} < \alpha < 2 \quad \text{for } k \geq 4. \quad (7)$$

*Proof.* We begin by computing the following chart for  $k \leq 5$ :

$k$	$2 - \frac{1}{k}$	$2 - \frac{1}{3k}$	$\alpha$
2	1.5	1.833...	1.618...
3	1.666...	1.889...	1.839...
4	1.75	1.916...	1.928...
5	1.8	1.933...	1.966...

It's clear that  $2 - \frac{1}{k} < \alpha < 2$  for  $2 \leq k \leq 5$  and that  $2 - \frac{1}{3k} < \alpha < 2$  for  $4 \leq k \leq 5$ . We now focus on  $k \geq 6$ . At this point, we could finish the proof by appealing to  $2(1 - 2^{-k}) < \alpha < 2$  as seen in the article [13, Lemma 3.6], but here we present a simpler proof.

Let  $f(x) = (x - 1)(x^k - x^{k-1} - \dots - x - 1) = x^{k+1} - 2x^k + 1$ . We know from our earlier discussion that  $f(x)$  has one real zero  $\alpha > 1$ . Writing  $f(x)$  as  $x^k(x - 2) + 1$ , we have

$$f\left(2 - \frac{1}{3k}\right) = \left(2 - \frac{1}{3k}\right)^k \left(\frac{-1}{3k}\right) + 1 \quad (8)$$

For  $k \geq 6$ , it's easy to show

$$3k < \left(\frac{5}{3}\right)^k = \left(2 - \frac{1}{3}\right)^k < \left(2 - \frac{1}{3k}\right)^k$$

Substituting this inequality into the right-hand side of (8), we can re-write (8) as

$$f\left(2 - \frac{1}{3k}\right) < (3k) \cdot \left(\frac{-1}{3k}\right) + 1 = 0.$$

Finally, we note that

$$f(2) = 2^{k+1} - 2 \cdot 2^k + 1 = 1 > 0,$$

so we can conclude that our root  $\alpha$  is within the desired bounds of  $2 - 1/3k$  and  $2$  for  $k \geq 6$ .  $\square$

We now have a lemma about the coefficients of  $\alpha^{n-1}$  in Theorems 1 and 2.

**Lemma 4.** *Let  $k \geq 2$  be an integer, and let  $m^{(k)}(x) = \frac{x - 1}{2 + (k + 1)(x - 2)}$ . Then,*

1.  $m^{(k)}(2 - 1/k) = 1$ .
2.  $m^{(k)}(2) = \frac{1}{2}$ .
3.  $m^{(k)}(x)$  is continuous and decreasing on the interval  $[2 - 1/k, \infty)$ .
4.  $m^{(k)}(x) > \frac{1}{x}$  on the interval  $(2 - 1/k, 2)$ .

*Proof.* Parts 1 and 2 are immediate. As for 3, note that we can rewrite  $m^{(k)}(x)$  as

$$m^{(k)}(x) = \frac{1}{k + 1} \left(1 + \frac{1 - \frac{2}{k+1}}{x - (2 - \frac{2}{k+1})}\right)$$

which is simply a scaled translation of the map  $y = 1/x$ . In particular, since this  $m^{(k)}(x)$  has a vertical asymptote at  $x = 2 - \frac{2}{k+1}$ , then by parts 1 and 2 we can conclude that  $m^{(k)}(x)$  is indeed continuous and decreasing on the desired interval.

To show part 4, we first note that in solving  $\frac{1}{x} = m^{(k)}(x)$ , we obtain a quadratic equation with the two intersection points  $x = 2$  and  $x = k$ . It's easy to show that  $\frac{1}{x} < m^{(k)}(x)$  at  $x = 2 - 1/k$ , and since both functions  $\frac{1}{x}$  and  $m^{(k)}(x)$  are continuous on the interval  $[2 - 1/k, \infty)$  and intersect only at  $x = 2$  and  $x = k \geq 2$ , we can conclude that  $\frac{1}{x} < m^{(k)}(x)$  on the desired interval.  $\square$

**Lemma 5.** For a fixed value of  $k \geq 2$  and for  $n \geq 2 - k$ , define  $E_n$  to be the error in our Binet approximation of Theorem 2, as follows:

$$\begin{aligned} E_n &= F_n^{(k)} - \frac{\alpha - 1}{2 + (k + 1)(\alpha - 2)} \cdot \alpha^{n-1} \\ &= F_n^{(k)} - m^{(k)}(\alpha) \cdot \alpha^{n-1}, \end{aligned}$$

for  $\alpha$  the positive real root of  $x^k - x^{k-1} - \dots - x - 1 = 0$  and  $m^{(k)}$  as defined in Lemma 4. Then,  $E_n$  satisfies the same recurrence relation as  $F_n^{(k)}$ :

$$E_n = E_{n-1} + E_{n-2} + \dots + E_{n-k} \quad (\text{for } n \geq 2).$$

*Proof.* By definition, we know that  $F_n^{(k)}$  satisfies the recurrence relation:

$$F_n^{(k)} = F_{n-1}^{(k)} + \dots + F_{n-k}^{(k)} \quad (9)$$

As for the term  $m^{(k)}(\alpha) \cdot \alpha^{n-1}$ , note that  $\alpha$  is a root of  $x^k - x^{k-1} - \dots - 1 = 0$ , which means that  $\alpha^k = \alpha^{k-1} + \dots + 1$ , which implies

$$m^{(k)}(\alpha) \cdot \alpha^{n-1} = m^{(k)}(\alpha) \alpha^{n-2} + \dots + m^{(k)}(\alpha) \alpha^{n-(k+1)} \quad (10)$$

We combine Equations (9) and (10) to obtain the desired result.  $\square$

## 4 Proof of Theorem 1

As mentioned above, Spickerman and Joyner [12] proved the following formula for the  $k$ -generalized Fibonacci numbers:

$$F_n^{(k)} = \sum_{i=1}^k \frac{\alpha_i^{k+1} - \alpha_i^k}{2\alpha_i^k - (k+1)} \alpha_i^{n-1} \quad (11)$$

Recall that the set  $\{\alpha_i\}$  is the set of roots of  $x^k - x^{k-1} - \dots - 1 = 0$ . We now show that this formula is equivalent to our Eq. (2) in Theorem 1:

$$F_n^{(k)} = \sum_{i=1}^k \frac{\alpha_i - 1}{2 + (k+1)(\alpha_i - 2)} \alpha_i^{n-1} \quad (12)$$

Since  $\alpha_i^k - \alpha_i^{k-1} - \dots - 1 = 0$ , we can multiply by  $\alpha_i - 1$  to get  $\alpha_i^{k+1} - 2\alpha_i^k = -1$ , which implies  $(\alpha_i - 2) = -1 \cdot \alpha_i^{-k}$ . We use this last equation to transform (12) as follows:

$$\frac{\alpha_i - 1}{2 + (k+1)(\alpha_i - 2)} = \frac{\alpha_i - 1}{2 + (k+1)(-\alpha_i^{-k})} = \frac{\alpha_i^{k+1} - \alpha_i^k}{2\alpha_i^k - (k+1)}$$

This establishes the equivalence of the two formulas (11) and (12), as desired.  $\square$

## 5 Proof of Theorem 2

Let  $E_n$  be as defined in Lemma 5. We wish to show that  $|E_n| < \frac{1}{2}$  for all  $n \geq 2 - k$ . We proceed by first showing that  $|E_n| < \frac{1}{2}$  for  $n = 0$ , then for  $n = -1, -2, -3, \dots, 2 - k$ , then for  $n = 1$ , and finally that this implies  $|E_n| < \frac{1}{2}$  for all  $n \geq 2 - k$ .

To begin, we note that since our initial conditions give us that  $F_n^{(k)} = 0$  for  $n = 0, -1, -2, \dots, 2 - k$ , then we need only show  $|m^{(k)}(\alpha) \cdot \alpha^{n-1}| < 1/2$  for those values of  $n$ . Starting with  $n = 0$ , it's easy to check by hand that  $m^{(k)}(\alpha) \cdot \alpha^{-1} < 1/2$  for  $k = 2$  and  $3$ , and as for  $k \geq 4$ , we have the following inequality from Lemma 3:

$$2 - \frac{1}{3k} < \alpha,$$

which implies

$$\alpha^{-1} < \frac{3k}{6k - 1}.$$

Also, by Lemma 4,

$$m^{(k)}(\alpha) < m^{(k)}(2 - 1/3k) = \frac{3k - 1}{5k - 1},$$

so thus:

$$m^{(k)}(\alpha) \cdot \alpha^{-1} < \frac{3k - 1}{5k - 1} \cdot \frac{3k}{6k - 1} < \frac{(3k) \cdot 1}{(5k - 1) \cdot 2} < \frac{1}{2},$$

as desired. Thus,  $0 < |m^{(k)}(\alpha) \cdot \alpha^{-1}| < 1/2$  for all  $k$ , as desired.

Since  $\alpha^{-1} < 1$ , we can conclude that for  $n = -1, -2, \dots, 2 - k$ , then  $|E_n| = m^{(k)}(\alpha) \cdot \alpha^{n-1} < 1/2$ .

Turning our attention now to  $E_1$ , we note that  $F_1^{(k)} = 1$  (again by definition of our initial conditions) and that

$$\frac{1}{2} = m(2) < m(\alpha) < m(2 - 1/k) = 1$$

which immediately gives us  $|E_1| < 1/2$ .

As for  $E_n$  with  $n \geq 2$ , we know from Lemma 5 that

$$E_n = E_{n-1} + E_{n-2} + \dots + E_{n-k} \quad (\text{for } n \geq 2)$$

Suppose for some  $n \geq 2$  that  $|E_n| \geq 1/2$ . Let  $n_0$  be the smallest positive such  $n$ . Now, subtracting the following two equations:

$$\begin{aligned} E_{n_0+1} &= E_{n_0} + E_{n_0-1} + \dots + E_{n_0-(k-1)} \\ E_{n_0} &= E_{n_0-1} + E_{n_0-2} + \dots + E_{n_0-k} \end{aligned}$$

gives us:

$$E_{n_0+1} = 2E_{n_0} - E_{n_0-k}$$

Since  $|E_{n_0}| \geq |E_{n_0-k}|$  (the first, by assumption, being larger than, and the second smaller than,  $1/2$ ), we can conclude that  $|E_{n_0+1}| > |E_{n_0}|$ . In fact, we can apply this argument repeatedly to show that  $|E_{n_0+i}| > \dots > |E_{n_0+1}| > |E_{n_0}|$ . However, this contradicts the observation from Eq. (3) that the error must eventually go to 0. We conclude that  $|E_n| < 1/2$  for all  $n \geq 2$ , and thus for all  $n \geq 2 - k$ .  $\square$

## 6 Acknowledgement

The first author would like to thank J. Siehler for inspiring this paper with his work on Tribonacci numbers.

## References

- [1] M. Cipu and F. Luca, On the Galois group of the generalized Fibonacci polynomial, *An. Ştiinţ. Univ. Ovidius Constanţa Ser. Mat.* **9** (2001), 27–38.
- [2] David E. Ferguson, An expression for generalized Fibonacci numbers, *Fibonacci Quart.* **4** (1966), 270–273.
- [3] I. Flores, Direct calculation of  $k$ -generalized Fibonacci numbers, *Fibonacci Quart.* **5** (1967), 259–266.
- [4] Hyman Gabai, Generalized Fibonacci  $k$ -sequences, *Fibonacci Quart.* **8** (1970), 31–38.
- [5] Dan Kalman, Generalized Fibonacci numbers by matrix methods, *Fibonacci Quart.* **20** (1982), 73–76.
- [6] David Kessler and Jeremy Schiff, A combinatoric proof and generalization of Ferguson’s formula for  $k$ -generalized Fibonacci numbers, *Fibonacci Quart.* **42** (2004), 266–273.
- [7] Gwang-Yeon Lee, Sang-Gu Lee, Jin-Soo Kim, and Hang-Kyun Shin, The Binet formula and representations of  $k$ -generalized Fibonacci numbers, *Fibonacci Quart.* **39** (2001), 158–164.
- [8] Claude Levesque, On  $m$ th order linear recurrences, *Fibonacci Quart.* **23** (1985), 290–293.
- [9] E. P. Miles, Jr., Generalized Fibonacci numbers and associated matrices, *Amer. Math. Monthly* **67** (1960), 745–752.
- [10] M. D. Miller, Mathematical Notes: On Generalized Fibonacci Numbers, *Amer. Math. Monthly* **78** (1971), 1108–1109.
- [11] W. R. Spickerman, Binet’s formula for the Tribonacci sequence, *Fibonacci Quart.* **20** (1982), 118–120.
- [12] W. R. Spickerman and R. N. Joyner, Binet’s formula for the recursive sequence of order  $k$ , *Fibonacci Quart.* **22** (1984), 327–331.
- [13] D. A. Wolfram, Solving generalized Fibonacci recurrences, *Fibonacci Quart.* **36** (1998), 129–145.



---

2000 *Mathematics Subject Classification*: Primary 11B39, Secondary 11C08, 33F05, 65D20.  
*Keywords*:  $k$ -generalized Fibonacci numbers, Binet, Tribonacci, Tetranacci, Pentanacci.

---

(Concerned with sequences [A000073](#), [A000078](#), and [A001591](#).)

---

Received February 23, 2014.

---