# A Combinatorial Proof of Vandermonde's Determinant

Arthur T. Benjamin and Gregory P. Dresden

We offer a combinatorial proof of Vandermonde's determinant

$$\begin{vmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ 1 & x_2 & x_2^2 & \dots & x_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{vmatrix} = \prod_{0 \le i < j \le n} (x_j - x_i)$$

that is as easy as playing cards.

Let $V_n$ denote the Vandermonde matrix with $(i, j)$ entry $v_{ij} = x_i^j$, for $0 \le i, j \le n$. Since the determinant of $V_n$ is a polynomial in $x_0, x_1, \dots, x_n$, it suffices to prove the identity for positive integers $x_0 \le x_1 \le \cdots \le x_n$. We define a *Vandermonde card* to possess a suit and a value, where a card of Suit $i$ has a value from the set $\{1, \dots, x_i\}$. (In our examples, we will let Suits $0, 1, 2, 3, 4$ be represented by suits $\odot, \clubsuit, \diamondsuit, \heartsuit, \spadesuit$, respectively.) Hence there are $x_0 + x_1 + \cdots + x_n$ different Vandermonde cards, but we have at our disposal an unlimited supply of each card to create *Vandermonde tables* described below. Now let's do some card counting.

**Card Counting Question 1:** How many ways can Vandermonde cards be arranged in $n+1$ rows, where Row 0 is empty, Row 1 has one card of Suit 1, Row 2 has two cards of Suit 2, Row 3 has three cards of Suit 3, ..., and Row $n$ has $n$ cards of Suit $n$? The order of the cards is important and we are allowed to repeat values of cards within each row. We call such an arrangement a *Vandermonde table associated with the identity permutation* $\pi = 012 \dots n$, an example of which is given in Figure 1.

| | Col 1 | Col 2 | Col 3 | Col 4 | | permutation $\pi$ |
|---|---|---|---|---|---|---|
| Row 0 | | | | | | $\pi(0) = 0 = \odot$ |
| Row 1 | $c_{11}\ \clubsuit$ | | | | $c_{11} \in \{1......x_1\}$ | $\pi(1) = 1 = \clubsuit$ |
| Row 2 | $c_{21}\ \diamondsuit$ | $c_{22}\ \diamondsuit$ | | | $c_{2j} \in \{1.........x_2\}$ | $\pi(2) = 2 = \diamondsuit$ |
| Row 3 | $c_{31}\ \heartsuit$ | $c_{32}\ \heartsuit$ | $c_{33}\ \heartsuit$ | | $c_{3j} \in \{1...........x_3\}$ | $\pi(3) = 3 = \heartsuit$ |
| Row 4 | $c_{41}\ \spadesuit$ | $c_{42}\ \spadesuit$ | $c_{43}\ \spadesuit$ | $c_{44}\ \spadesuit$ | $c_{4j} \in \{1..............x_4\}$ | $\pi(4) = 4 = \spadesuit$ |

Figure 1: A Vandermonde table associated with the identity permutation $\pi = 01234$ (or $\pi = \odot\clubsuit\diamondsuit\heartsuit\spadesuit$). Each of the $i$ cards in Row $i$ has Suit $i$ and a value from $\{1, \dots, x_i\}$. Such a table can be created $x_1 x_2^2 x_3^3 x_4^4$ ways.

**Answer:** For $0 \le i \le n$, the $i$ cards in Row $i$ all have Suit $i$, so their values can be assigned $x_i^i$ ways. Hence, the number of arrangements is $1 x_1 x_2^2 x_3^3 \cdots x_n^n$, which is the product of the diagonal entries of $V_n$.

**Card Counting Question 2:** Same as Question 1, but now we are given a permutation $\pi$ of the numbers 0 through $n$, say $\pi = a_0 a_1 \ldots a_n$. Here, Row $i$ must contain $i$ cards from Suit $\pi(i) = a_i$. We call such an arrangement a *Vandermonde table with permutation* $\pi$. A typical table is shown in Figure 2.

**Answer:** Counting row by row again, there are $1x^1_{\pi(1)}x^2_{\pi(2)}x^3_{\pi(3)}\cdots x^n_{\pi(n)}$ such tables, which is the product of the $n+1$ entries of the form $v_{\pi(i),i}$ from $V_n$.

| | Col 1 | Col 2 | Col 3 | Col 4 | | permutation $\pi$ |
|---|---|---|---|---|---|---|
| Row 0 | | | | | | $\pi(0) = 3 = \heartsuit$ |
| Row 1 | $c_{11}$ ♠ | | | | $c_{11} \in \{1 \ldots\ldots\ldots\ldots x_4\}$ | $\pi(1) = 4 =$ ♠ |
| Row 2 | $c_{21}$ $\odot$ | $c_{22}$ $\odot$ | | | $c_{2j} \in \{1 \ldots x_0\}$ | $\pi(2) = 0 = \odot$ |
| Row 3 | $c_{31}$ $\diamondsuit$ | $c_{32}$ $\diamondsuit$ | $c_{33}$ $\diamondsuit$ | | $c_{3j} \in \{1 \ldots\ldots\ldots x_2\}$ | $\pi(3) = 2 = \diamondsuit$ |
| Row 4 | $c_{41}$ $\clubsuit$ | $c_{42}$ $\clubsuit$ | $c_{43}$ $\clubsuit$ | $c_{44}$ $\clubsuit$ | $c_{4j} \in \{1 \ldots\ldots x_1\}$ | $\pi(4) = 1 = \clubsuit$ |

Figure 2: A Vandermonde table associated with permutation $\pi = 34021$ (or $\pi = \heartsuit$♠$\odot\diamondsuit\clubsuit$). Each of the $i$ cards in Row $i$ has Suit $\pi(i)$ and a value from $\{1, \ldots, x_{\pi(i)}\}$. Such a table can be created $x_4 x_0^2 x_2^3 x_1^4$ ways.

**Card Counting Question 3:** Same as Question 2, but now $\pi$ is not prescribed in advance, so $\pi$ can be any permutation of $\{0, \ldots, n\}$. As before, each row is assigned a different suit and each Row $i$ contains $i$ cards of the assigned suit. For this unrestricted problem, such an arrangement is simply called a *Vandermonde table*.

**Answer:** Sum the answer to Question 2 over all possible permutations of $0, \ldots, n$. In other words, the number of ways to create a Vandermonde table is the *permanent* of $V_n$.

**Card Counting Question 4:** Same as Question 3, but now we count those arrangements with even permutations positively and those arrangements with odd permutations negatively.

**Answer:** By definition, this is the *determinant* of $V_n$.

It remains to show that the answer to Question 4 also equals $\prod_{0 \le i < j \le n}(x_j - x_i)$.

For a given Vandermonde table $C$, let the cards of Row $i$ be denoted by $C_{i1}, C_{i2}, \ldots, C_{ii}$, with values $c_{i1}, c_{i2}, \ldots, c_{ii}$. We say that card $C_{ij}$ is *small* if $c_{ij} \le x_{j-1}$. For example, any card in Column 1 with a value less than or equal to $x_0$ (such as any card of Suit 0) is small.

**Card Counting Question 5:** How many Vandermonde tables have no small cards?

**Answer:** Let $C$ be a Vandermonde table with no small cards. Since Column 1 must not contain any cards of Suit 0, Suit 0 must be assigned to the empty Row 0. To avoid small cards in Column 2, card $C_{11}$ must be given Suit 1 (since all cards of Suit 1 have value less than or equal to $x_1$), so Row 1 must be assigned Suit 1. By the same reasoning, Row 2 must have Suit 2, …, and Row $n$ must have Suit $n$. Thus $C$ must be associated with the identity permutation. Furthermore, to avoid small cards in the first column, the values of the cards $C_{11}, \ldots, C_{n1}$ can be assigned $(x_1 - x_0)(x_2 - x_0)(x_3 - x_0)\cdots(x_n - x_0)$ ways. Likewise, the values of the cards in the second column can be assigned $(x_2 - x_1)(x_3 - x_1)\cdots(x_n - x_1)$ ways, and so on down to the single card of Suit $n$ in the last column, with a value that can be assigned $x_n - x_{n-1}$ ways. Hence there are $\prod_{0\le i<j\le n}(x_j - x_i)$ Vandermonde tables with no small cards.

We say that a Vandermonde table is *good* if it has no small cards, and is *bad* if it has at least one small card. Note that since the identity permutation is even, all of the good tables are counted *positively* in the determinant of $V_n$.

To complete the proof of Vandermonde's determinant, it suffices to show that every bad Vandermonde table can be paired up with another bad Vandermonde table with a permutation of opposite parity. Thus, when the determinant of $V_n$ sums over all Vandermonde tables, the bad tables will cancel each other out. When the dust settles, only the good tables (all counted positively) will remain standing.

Now let $C$ be a bad Vandermonde table with permutation $\pi = a_0a_1\dots a_n$. We define the *first small card* of $C$ to be the small card $c_{ij}$ where $j$ is as small as possible, and if Column $j$ has more than one small card then we choose $i$ to be as large as possible. In other words, we look for small cards from bottom to top, beginning in Column 1.

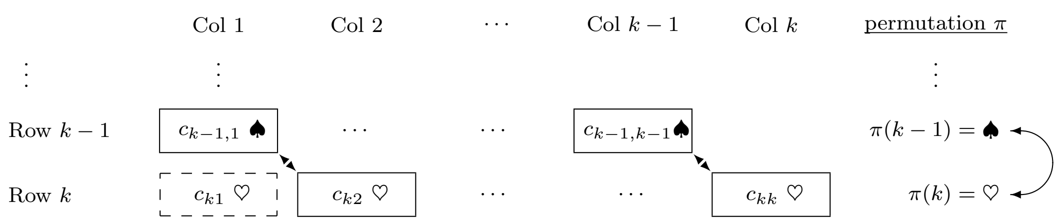


Figure 3: When the first small card occurs in the first column at card $C_{k1}$, simply swap the cards of Row $k-1$ with the cards $C_{k2}, \dots, C_{kk}$, and change the suit of card $C_{k1}$.

Suppose that the first small card of $C$ occurs in Column 1, say card $C_{k1}$ for some $1 \le k \le n$. Then, since $C_{k1}$ is small, $c_{k1} \le x_0$ and since it is the first small card, there are no small cards below it; that is, for $i > k$, $c_{i1} > x_0$. For definiteness, suppose that the cards in Row $k-1$ have Suit $\pi(k-1) =$ ♠ and the cards in Row $k$ have Suit $\pi(k) =$ ♡. (We make no assumptions about the suit number for hearts or spades.) Now consider the Vandermonde table $C'$ obtained by swapping all $k-1$ spade cards with all of the heart cards except for card $C_{k1}$. Then change the suit of card $C_{k1}$ from hearts to spades. The suit change from hearts to spaces is legal since $C_{k1}$ has value $c_{k1} \le x_0$, which is a legal value for all suits. (Here we are exploiting the fact that $x_0 \le x_1 \le \cdots \le x_n$.) Notice that $C_{k1}$ is still the first small card of $C'$, albeit with a new suit, and thus if we apply the swapping procedure to $C'$, we obtain $C$. That is, $(C')' = C$. Furthermore, $C'$ has permutation $\pi' = a_0a_1\dots a_ka_{k-1}\dots a_n$. Permutations $\pi$ and $\pi'$ have opposite parity since they differ by the transposition of hearts and spades. See Figure 3.

Now suppose that the first small card of $C$ occurs in Column $j$ for some $j \ge 2$, say at card $C_{kj}$. Consequently, $c_{kj} \le x_{j-1}$, and there are no small cards anywhere in Columns 1 through $j-1$ nor below card $C_{kj}$ in Column $j$. As before, suppose the cards of Row $k$ have the heart suit, and that the cards of Row $k-1$ have the spade suit. Create a new Vandermonde table $C'$ by swapping the first $j-1$ cards of Rows $k-1$ and $k$, leaving card $C_{kj}$ in its place, but changing its suit from hearts to spades, then swapping the remaining $k-j$ cards of Rows

$k-1$ and $k$, as in Figure 4.

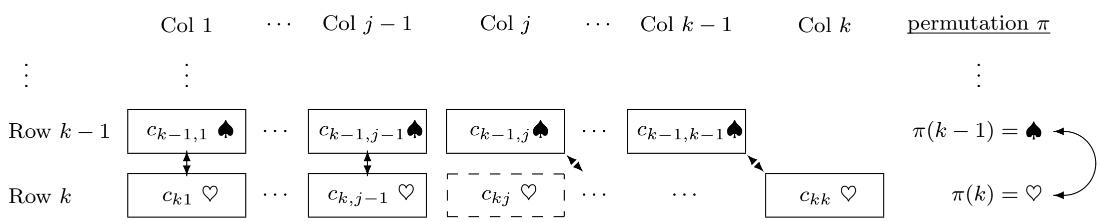


Figure 4: When $C_{kj}$ is the first small card, and $j \geq 2$, then swap the first $j-1$ cards of Row $k-1$ with the first $j-1$ cards of Row $k$, change the suit of card $C_{kj}$, then swap the remaining cards of Rows $k-1$ and $k$. In the new Vandermonde table, card $C_{kj}$ remains the first small card.

Why is it legal to change the suit of card $C_{kj}$ from hearts to spades? Since $C_{kj}$ was the first small card, then the spade card $C_{k-1,j-1}$ is not small and therefore has a value strictly greater than $x_{j-2}$. Thus all spade cards can take on values less than or equal to $x_{j-1}$. Since $C_{kj}$ is small, its value is at most $x_{j-1}$, so changing it from hearts to spades is allowable.

As before, $C_{kj}$ remains the first small card of $C'$, so $(C')' = C$ and $C'$ has permutation $\pi'$, which has opposite parity of $\pi$ since they differ by a transposition. Thus there is a 1-1 correspondence between the positively counted Vandermonde tables with small cards and the negatively counted Vandermonde tables with small cards. Therefore the determinant of $V_n$ is the number of Vandermonde tables with no small cards, namely $\prod_{0 \leq i < j \leq n}(x_j - x_i)$, as desired.

For another combinatorial proof of Vandermonde's determinant, where the cancellation occurs in the product instead of the sums, see the short paper by Ira Gessel [1].

## References


[1] I. Gessel, Tournaments and Vandermonde's Determinant, *Journal of Graph Theory* **3** (1979) 305–307.



*Harvey Mudd College*
*Claremont, CA 91711*
`benjamin@math.hmc.edu`

*Washington and Lee University*
*Lexington, VA 24450*
`dresdeng@wlu.edu`