

Three Transcendental Numbers From the Last Non-Zero Digits of n^n , F_n , and $n!$.

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In this article, we will construct three infinite decimals from the last nonzero digits of n^n , F_n (the Fibonacci numbers), and $n!$, respectively, and we will show that all three are transcendental. Along the way, we will learn a bit about the history of transcendental numbers, discuss two major theorems in the field, and pose some questions for future research. Let's begin by recalling what it means for a number to be transcendental.

DEFINITION. For α a complex number, we say α is **algebraic** if it is the root of a polynomial with integer coefficients. If no such polynomial exists, we say that α is **transcendental**.

Early mathematicians, of course, were ignorant of this distinction. Indeed, the Pythagoreans of ancient Greece believed that everything in the universe could be measured by whole numbers and their ratios. It must have come as quite a shock when Hippasus (fifth century B.C.E.) first demonstrated that certain ratios, such as the ratio between the diagonal and the side of a square, were not the ratios of two whole numbers. Legend has it that angry Pythagoreans threw Hippasus into the sea for his heresy [9], but the idea of irrational numbers lived on. Indeed, our word “irrational” dates back to the Greek word $\alpha\rho\rho\eta\tau\omicron\varsigma$ (*arrētos*), meaning “unspeakable”, perhaps reflecting the Greeks’ disgust of such messy objects as $\sqrt{2}/1$ [9], [15].

We now skip ahead to seventeenth-century Europe. With the introduction of algebra and modern notation, it finally became possible to ask if there

existed numbers that were not roots of polynomials. James Gregory [3],[8] appears to have been the first to attempt to prove that π and e were not algebraic, an ambitious (and ultimately fruitless) goal given that it was not yet known if they were even irrational (their irrationality was finally shown in the eighteenth century by Lambert and Euler, respectively).

It was not until the nineteenth century that we see the first proofs of the existence of transcendental numbers. Liouville was the first to do so, with the specially-constructed number

$$L = \sum_{n=1}^{\infty} \frac{1}{10^{n!}} = 0.11000100000000000000000001\dots$$

The numbers π and e were shown to be transcendental by the later part of the century by Lindemann and Hermite, respectively. Lindemann's proof finally put to rest the old problem of squaring the circle, first studied by the Greeks over two millenia earlier. Lindemann later reported [12, p. 246] that Kronecker said to him (probably in jest, and perhaps alluding to the ancient Greeks' distaste for irrational numbers), "Of what use is your beautiful investigation of π ? Why study such problems since irrational numbers do not exist?" Some have seen this as a demonstration that Kronecker believed only in the existence of integers (recall also his famous quote "God created the integers, all else is the work of man"), but it is clear from his work that this is not the case. For a summary of the controversy over Kronecker's words, see the article by Edwards [5, Essay 5.5].

We note that even in the twenty-first century there are many open questions about transcendental numbers. It's not known if $\zeta(3)$ is transcendental, and as for the Euler constant γ , we do not yet know if it is even irrational. For a further discussion of recent work on transcendental numbers, Ribenboim in [13] provides an English summary of a 1983 historical overview by Waldschmidt [16]. See also Ribenboim's extensive bibliography in [13, Chapter 10].

Two Modern Theorems

It is possible to prove that certain numbers (such as π and e) are transcendental by assuming them to be algebraic and then working towards a contradiction. Such proofs are fairly complicated and involve a lot of auxiliary polynomials [7], [2]. Fortunately, there exist several different characterizations of transcendental numbers that will prove to be much easier to work with. This first theorem is the result of the work of three mathematicians over the first half of the twentieth century; it is restated slightly for convenience.

THEOREM 1 (THUE, SIEGEL, ROTH). *For α algebraic and $\epsilon > 0$, then there exist only finitely many rational numbers p/q such that*

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^{2+\epsilon}}.$$

Thus, if we can show that for some fixed $\epsilon > 0$ there are infinitely many such numbers p/q that satisfy the above inequality, then α must be transcendental. In contrast to Thue-Siegel-Roth, consider the following fairly trivial result from Diophantine approximation [7], [11]: if we have α rational and $\epsilon > 0$ then there exist only finitely many many rational numbers p/q such that $\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^{1+\epsilon}}$. Note also that both these results are best possible in terms of the exponents. That is, if we replace the epsilons with zeros, then there would actually be infinitely many p/q 's (found by continued fractions) satisfying the first inequality and infinitely many p/q 's (for any q , take $p = \lfloor \alpha q \rfloor$) for the second inequality.

Our second theorem comes from an article just published in 2004, and it is an extremely useful result. The following presentation is translated from the original French and is simplified for convenience.

THEOREM 2 (ADAMCZEWSKI, BUGEAUD, LUCA). *Let α be irrational, and suppose there exist two sequences $\{U_n\}$, $\{V_n\}$ of finite words on $\{0, 1, \dots, 9\}$ and a real number $x > 1$ such that*

- (1) For every $n \geq 1$, the word $U_n V_n^x$ is a prefix for α ,
- (2) The set $\left(\frac{|U_n|}{|V_n|}\right)_{n \geq 1}$ is bounded,
- (3) The set $|V_n|$ is strictly increasing.

Then, α is transcendental.

A few comments and definitions are in order. Recall that a **finite word** W is simply a collection of digits, such as 411 or 314159. In saying that a word W is a **prefix** for a number α we mean that α begins with the digits from W . By $|W|$ we mean the number of digits in W . For x an integer, then W^x is simply the word $WW \cdots W$ (repeated x times), while if x is not integral we define W^x as being $W^{\lfloor x \rfloor}$ followed by the first $\lceil (x - \lfloor x \rfloor) \cdot |W| \rceil$ elements of W . In particular, if W is 411, then $W^{1.5}$ would be 41141. Finally, the original statement and proof of this theorem in [1] is slightly more general in that it holds for α expressed in any base.

Both of these theorems give us new ways of identifying transcendental numbers. The Thue-Siegel-Roth theorem requires us to find an extremely close rational approximation, while the Adamczewski-Bugeaud-Luca theorem relies on a pattern in the digits. Let us now move on to our three transcendental numbers and see how we can apply these two theorems.

Forming numbers from the digits of n^n

We begin by looking at the pattern formed from the last (i.e. unit) digit of n^n . Since

$$1^1 = \boxed{1}, \quad 2^2 = \boxed{4}, \quad 3^3 = 2\boxed{7}, \quad 4^4 = 25\boxed{6}, \quad 5^5 = 312\boxed{5}, \quad \dots$$

then if we take the last digit of each number and form a decimal, we get

$$\begin{aligned} 0. & 1476563690 \ 1636567490 \\ & 1476563690 \ 1636567490 \ \dots \end{aligned}$$

This looks a lot like a repeating decimal, and indeed it is not hard to prove (see [6]) that

$$n^n \equiv (n + 20k)^{n+20k} \pmod{10}$$

which allows us to conclude that we have a rational number with period 20 equal to $(1476563690\ 1636567490) / (9999999999\ 9999999999)$. Since rational numbers are by definition not transcendental, we need to modify our construction to produce a more interesting decimal.

With this in mind, let us now construct a new decimal number $A = 0.d_1d_2d_3 \dots d_n \dots$ such that the n^{th} digit d_n of A is the last nonzero digit of n^n ; that is,

$$A = 0. \ 1476563691\ 1636567496 \\ 1476563699\ 1636567496 \ \dots$$

The reader will note that this new number A differs from the previous number at the tenth decimal place, the twentieth, the thirtieth, and so on. In a recent paper [4], we showed that this A is an irrational number (despite “almost” having a period of twenty digits). We will now show that it is transcendental.

To prove transcendence, we will be using the Thue-Siegel-Roth Theorem. In particular, we will demonstrate the existence of an infinite sequence of rational numbers p_n/q_n such that $\left|A - \frac{p_n}{q_n}\right| < \frac{1}{q_n^{2.1}}$.

Let’s begin by creating a sequence of irrational numbers $A_0 = A, A_1, A_2, \dots$ such that each A_n is formed by replacing every digit in A_{n-1} with zeros except for every tenth nonzero digit (which will be left alone). This means that each A_n has nonzero digits only every 10^n th place, at $10^{-10^n}, 10^{-2 \cdot 10^n}, 10^{-3 \cdot 10^n}$, and so on. Visually, this sequence looks like the following, where the dots represent zeros.

$$A_0 = 0.1476563691\ 1636567496\ 1476563699\ 1636567496 \ \dots \\ A_1 = 0.\dots\dots\dots 1 \ \dots\dots\dots 6 \ \dots\dots\dots 9 \ \dots\dots\dots 6 \ \dots \\ A_2 = 0.\dots\dots\dots \dots\dots\dots \dots\dots\dots \dots\dots\dots \dots\dots\dots \dots\dots \dots$$

(The number A_2 doesn't have a nonzero digit until the one hundredth decimal place, at 10^{-100} .) If we remove the dots and condense these decimals a bit, we see an interesting pattern develop (recall that the nonzero digits in A_n are actually 10^n decimal places apart; think of the spaces in A_1, A_2 , etc. as representing lots and lots of zeros)

$$\begin{aligned} A_0 &= 0.1476563691 \dots \\ A_1 &= 0. \quad 1 \quad 6 \quad 9 \quad 6 \quad 5 \quad 6 \quad 9 \quad 6 \quad 1 \quad 1 \quad \dots \\ A_2 &= 0. \quad 1 \quad 6 \quad 1 \quad 6 \quad 5 \quad 6 \quad 1 \quad 6 \quad 1 \quad 1 \quad \dots \\ A_3 &= 0. \quad 1 \quad 6 \quad 1 \quad 6 \quad 5 \quad 6 \quad 1 \quad 6 \quad 1 \quad 1 \quad \dots \end{aligned}$$

A simple application of [4, Lemma 3] shows that for $n \geq 2$, the sequences of nonzero digits in each A_n are identical: 1, 6, 1, 6, 5, 6, 1, 6, 1, *, where * is either 1, 6, or 5 depending on the position. This implies that $R_n = A_n - A_{n+1}$ is rational for $n \geq 2$; the cases $n = 0$ and $n = 1$ follow immediately from Lemma 2 below. Since each R_n is rational, then if we can show that A_n is well approximated by rationals (in the context of the Thue-Siegel-Roth theorem), this might help us to approximate A as well. Let's investigate these A_n 's a bit further.

If we write out just the nonzero digits of A_n (with appropriate spacing)

$$A_n = 0. \quad 1 \quad 6 \quad 1 \quad 6 \quad 5 \quad 6 \quad 1 \quad 6 \quad 1 \quad 1 \quad \dots \quad (\text{for } n \geq 2)$$

we clearly see that each A_n (for $n \geq 2$) is quite close to the rational number

$$\frac{s_n}{t_n} = 0. \quad 1 \quad 6 \quad 1 \quad 6 \quad 1 \quad 6 \quad 1 \quad 6 \quad 1 \quad 6 \quad \dots$$

Here, just as with A_n , these nonzero digits of s_n/t_n are actually 10^n decimal places apart. Thus, $\frac{s_n}{t_n}$ is easily seen to be $\frac{1 \cdot 10^{10^n} + 6}{10^{2 \cdot 10^n} - 1}$. Also, it is easy to see that A_n and s_n/t_n differ in the fifth visible position (among other places), which means that they differ by about $\frac{4}{10^{5 \cdot 10^n}}$. As a result, we have

$$\left| A_n - \frac{s_n}{t_n} \right| \approx \frac{4}{10^{5 \cdot 10^n}} < \frac{1}{10^{4.2 \cdot 10^n}} \approx \frac{1}{t_n^{2.1}}$$

(The attentive reader will notice that we could have easily replaced the 2.1 with 2.4 or even 2.49. However, 2.1 will work fine for our purposes.)

Let's now relate this back to A . Recall that we have $A_{n+1} = A_n - R_n$, and it is easy to show that the rational number R_n has denominator $10^{10^{n+1}} - 1$ for $n > 0$ and denominator $10^{20} - 1$ at $n = 0$. We define p_n/q_n as

$$\frac{p_n}{q_n} = \left(\sum_{i=0}^{n-1} R_i \right) + \frac{s_n}{t_n}.$$

The denominator q_n is $10^{2 \cdot 10^n}$, the same as t_n , because the denominator of each R_i divides evenly into t_n . Thus,

$$\left| A - \frac{p_n}{q_n} \right| = \left| A - \sum_{i=0}^{n-1} R_i - \frac{s_n}{t_n} \right| = \left| A_n - \frac{s_n}{t_n} \right| \approx \frac{4}{10^{5 \cdot 10^n}} < \frac{1}{10^{4.2 \cdot 10^n}} \approx \frac{1}{t_n^{2.1}} = \frac{1}{q_n^{2.1}}$$

The conditions of Theorem 1 being satisfied, we can conclude that A is transcendental.

A Number from the Fibonacci Sequence

Let us look at a new decimal number, this one constructed from the Fibonacci sequence F_n . It's easy to show that $F_{n+60} \equiv F_n \pmod{10}$, which means that as in the case of n^n , taking the last digit of each number would result in a repeating decimal. Instead, we construct a decimal (call it B) from the last nonzero digits. Since the Fibonacci numbers are

$\boxed{1}, \boxed{1}, \boxed{2}, \boxed{3}, \boxed{5}, \boxed{8}, 1\boxed{3}, 2\boxed{1}, 3\boxed{4}, 5\boxed{5}, 8\boxed{9}, 14\boxed{4}, 23\boxed{3}, \dots$

then we get

$$\begin{aligned} B = & 0.112358314594371 \ 774156178538194 \\ & 998752796516737 \ 336954932572912 \\ & 112358314594375 \ 774156178538192 \ \dots \end{aligned}$$

Our technique in proving that B is transcendental is the same as our proof earlier with A (formed from the last nonzero digit of n^n). We will construct

a sequence of decimals $B_0 = B, B_1, B_2$, each formed by selecting a few digits from the previous number. Each B_n will be well approximated by a rational number s_n/t_n , which will be used to create another rational number p_n/q_n . Using Theorem 1 and these rational numbers p_n/q_n , we will be able to conclude that B is transcendental.

Let's begin with B_1 , which will be formed from B_0 by keeping every fifteenth digit in B_0 , and replacing all the others with zeros.

$$\begin{aligned}
 B_1 = 0. & \dots\dots\dots 1 \dots\dots\dots 4 \\
 & \dots\dots\dots 7 \dots\dots\dots 2 \\
 & \dots\dots\dots 5 \dots\dots\dots 2 \dots
 \end{aligned}$$

We condense and continue B_1 to get

$$\begin{aligned}
 B_1 = 0. & \ 1 \ 4 \ 7 \ 2 \ 5 \ 2 \ 3 \ 4 \ 9 \ 2 \quad 9 \ 6 \ 3 \ 8 \ 5 \ 8 \ 7 \ 6 \ 1 \ 6 \\
 & \quad 1 \ 4 \ 7 \ 2 \ 5 \ 2 \ 3 \ 4 \ 9 \ 6 \quad 9 \ 6 \ 3 \ 8 \ 5 \ 8 \ 7 \ 6 \ 1 \ 2 \quad \dots
 \end{aligned}$$

(digits are 15 decimal places apart, separated by a stream of 0's).

Now define B_2 to be every tenth nonzero digit of B_1 , with zeros elsewhere.

$$\begin{aligned}
 B_2 = 0. & \ 2 \ 6 \ 6 \ 2 \ 1 \quad 8 \ 4 \ 4 \ 8 \ 8 \\
 & \quad 2 \ 6 \ 6 \ 2 \ 3 \quad 8 \ 4 \ 4 \ 8 \ 6 \quad \dots
 \end{aligned}$$

(digits are 150 decimal places apart, separated by zeros).

Define B_3 to be every fifth nonzero digit of B_2 .

$$\begin{aligned}
 B_3 = 0. & \ 1 \ 8 \ 3 \ 6 \ 5 \ 4 \ 7 \ 2 \ 9 \ 9 \\
 & \quad 1 \ 8 \ 3 \ 6 \ 5 \ 4 \ 7 \ 2 \ 9 \ 8 \quad \dots
 \end{aligned}$$

(digits are 750 decimal places apart).

Let B_4 be every tenth nonzero digit of B_3 , and we will finally see a beautiful pattern appear.

$$\begin{aligned}
 B_4 = 0. & \ 9 \ 8 \ 7 \ 6 \ 5 \ 4 \ 3 \ 2 \ 1 \ 9 \\
 & \quad 9 \ 8 \ 7 \ 6 \ 5 \ 4 \ 3 \ 2 \ 1 \ 8 \quad \dots
 \end{aligned}$$

(digits are now 7500 decimal places apart).

If we define B_n (for $n \geq 4$) to be every tenth nonzero digit of B_{n-1} , we will always get (Lemma 4) the pattern

$$B_n = 0. \ 9 \ 8 \ 7 \ 6 \ 5 \ 4 \ 3 \ 2 \ 1 \ 9 \\ \qquad \qquad \qquad 9 \ 8 \ 7 \ 6 \ 5 \ 4 \ 3 \ 2 \ 1 \ 8 \ \dots \quad (\text{for } n \geq 4)$$

(digits are $75 \cdot 10^{n-2}$ decimal places apart). Furthermore, these B_n 's have all been chosen such that they differ from their predecessors by a rational number; that is, if we define $R_n = B_{n+1} - B_n$, then R_n is rational.

Consider the rational number $\frac{s_n}{t_n} = \frac{9 \cdot 10^{75 \cdot 10^{n-2}} - 10}{(10^{75 \cdot 10^{n-2}} - 1)^2}$. Writing out the first few nonzero digits, we find that this rational number is

$$\frac{s_n}{t_n} = 0. \ 9 \ 8 \ 7 \ 6 \ 5 \ 4 \ 3 \ 2 \ \dots .$$

After that 2, the decimal expansion becomes rather complicated, with most of the zeros replaced by nines, and then some eights start to show up, but that doesn't really concern us. What's important is that s_n/t_n is a good approximation to B_n for quite a few decimal places; in fact, they are the same up to the eighth nonzero digit. Thus, for $n \geq 4$,

$$\left| B_n - \frac{s_n}{t_n} \right| < \frac{1}{10^{8 \cdot (75 \cdot 10^{n-2})}} \approx \frac{1}{t_n^4}$$

Let's now relate this back to B_4 . If we can show that B_4 is transcendental, then since B_4 and B differ by a rational number, then B itself will also be transcendental.

Recall that we have $B_{n+1} = B_n - R_n$, and it is easy to show that R_i has denominator $10^{75 \cdot 10^{i-1}} - 1$ (for $i \geq 4$). We now define p_n/q_n as

$$\frac{p_n}{q_n} = \left(\sum_{i=4}^{n-1} R_i \right) + \frac{s_n}{t_n}.$$

The denominator q_n is $(10^{75 \cdot 10^{n-2}} - 1)^2$, the same as t_n , because the denominator of each R_i divides evenly into t_n . Thus,

$$\left| B_4 - \frac{p_n}{q_n} \right| = \left| B_4 - \sum_{i=4}^{n-1} R_i - \frac{s_n}{t_n} \right| = \left| B_n - \frac{s_n}{t_n} \right| < \frac{1}{10^{8 \cdot (75 \cdot 10^{n-2})}} \approx \frac{1}{t_n^4}$$

Thus, by Theorem 1, B_4 is transcendental, and hence so is B .

A Number from the Sequence $n!$

Finally, let us look at what we get from the last nonzero digit of $n!$. Starting with

$$\boxed{1}, \boxed{2}, \boxed{6}, \boxed{24}, \boxed{120}, \boxed{720}, \boxed{5040}, \boxed{40320}, \boxed{362880}, \boxed{3628800}, \dots$$

we get

$$C = 0.1264 \ 22428 \ 88682 \ 88682 \ 44846 \ 44846 \ 88682 \ \dots$$

It looks more appealing if we replace the first block of 1264 with 66264, and continue out a few thousand digits. We then get the very nice pattern

$$\begin{array}{r} C' = 0.66264 \ 22428 \ 88682 \ 88682 \ 44846 \ \dots \quad (625 \text{ digits per line}) \\ \quad 66264 \ 22428 \ 88682 \ 88682 \ 44846 \ \dots \\ \quad 22428 \ 44846 \ 66264 \ 66264 \ 88682 \ \dots \\ \quad 66264 \ 22428 \ 88682 \ 88682 \ 44846 \ \dots \\ \quad 44846 \ 88682 \ 22428 \ 22428 \ 66264 \ \dots \\ \\ \quad 22428 \ 44846 \ 66264 \ 66264 \ 88682 \ \dots \\ \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \end{array}$$

Note that C and C' are related by the linear formula $C = 10C' - 6.5$, so if C' is identified as rational, algebraic, or transcendental, then the same must hold for C itself. The number C' turns out to be much easier to work with (notice the nice repetition in the first two lines, for example), and with this in mind, let us define ℓ to represent the last nonzero digit with a slight variation.

$$\text{DEFINITION. For } n \geq 0, \text{ define } \ell(n) = \begin{cases} 6 & : n = 0, 1 \\ \text{lnzd}(n!) & : n \geq 2. \end{cases}$$

In a previous paper [4], we proved that C (and thus C') is irrational. Thanks to the new Adamczewski-Bugeaud-Luca theorem, we can now show that it is transcendental. An important step in this direction is the following formula for the decimal digits in C' . Writing n in base-5 notation as $n = \sum_{i=0}^N a_i 5^i$ (for $a_i \in \{0, 1, 2, 3, 4\}$), then $\ell(n) = 6 \prod_{i=0}^N (a_i!) 2^{i \cdot a_i} \pmod{10}$ (see Lemma 7). We can use this to explain why the first two lines in the expression for C' are identical, and in fact we can go a bit further. The following lemma gives the full story.

LEMMA 1. *For r a multiple of four, the first 5^r digits of C' equal the second 5^r digits.*

Proof. Let n be a positive integer less than 5^r . We can write $n = \sum_{i=0}^{r-1} a_i 5^i$, and thus $n + 5^r = \sum_{i=0}^{r-1} (a_i 5^i) + 1 \cdot 5^r$. So, by our formula for $\ell(n)$, we have $\ell(n + 5^r) = \ell(n) \cdot (1!) 2^{r \cdot 1} \pmod{10}$. Since r is a multiple of 4, then $2^r \equiv 6 \pmod{10}$, and recalling that 6 acts as an identity on the set $\{2, 4, 6, 8\}$ under multiplication mod 10, we have $\ell(n + 5^r) = \ell(n)$. ■

We now use this lemma to immediately prove the transcendence of the irrational number C' . In the notation of the Adamczewski-Bugeaud-Luca theorem, let $U_n = \{\}$, let V_n be the first 5^{4n} digits of C' , and let $x = 2$. All three criteria of Theorem 2 having been satisfied, we conclude that C' , and hence C , is transcendental.

Proofs

The following technical lemmas are used in the paper.

LEMMA 2. *For n not divisible by 100, then*

$$\text{lnzd}(n^n) = \text{lnzd}((n + 100)^{n+100}).$$

Proof. We first note that if $100 \nmid n$ then $\text{lnzd}(n) = \text{lnzd}(n + 100)$. Also, $\text{lnzd}(a^b) = \text{lnzd}(a)^b \pmod{10}$ for all $a, b > 0$. This allows us to state that $\text{lnzd}((n + 100)^{n+100}) = \text{lnzd}(n + 100)^{n+100} \pmod{10} = \text{lnzd}(n)^{n+100} \pmod{10}$, so the lemma reduces to proving that $\text{lnzd}(n)^{n+100} \equiv \text{lnzd}(n)^n \pmod{10}$. This is clearly true for $\text{lnzd}(n) = 5$. If $\text{lnzd}(n) = 1, 3, 7$, or 9 , then since each of these raised to the fourth power gives $1 \pmod{10}$, we have $\text{lnzd}(n)^{n+100} \equiv \text{lnzd}(n)^n \cdot 1^{25} = \text{lnzd}(n)^n$. If on the other hand $\text{lnzd}(n) = 2, 4, 6$, or 8 , then since each of these gives 6 when raised to the fourth power mod 10 , we have $\text{lnzd}(n)^{n+100} \equiv \text{lnzd}(n)^n \cdot 6^{25} \equiv \text{lnzd}(n)^n \cdot 6$. Now, $\text{lnzd}(n)$ is even, and 6 acts as a multiplicative identity on even numbers mod 10 , so we end up with $\text{lnzd}(n)^n$, as desired. ■

LEMMA 3. For $n = 7500 \cdot 10^k$, $k \geq 0$, then $\text{lnzd}(F_n) = 9$.

Proof. This is certainly true for $k = 0$, by inspection. For $k > 0$ we use induction and the identity [18, Formula 42]

$$F_{10n} = F_n \left[\binom{9}{0} L_n^9 - \binom{8}{1} L_n^7 + \binom{7}{2} L_n^5 - \binom{6}{3} L_n^3 + \binom{5}{4} L_n \right].$$

Assume that n is as given and that $\text{lnzd}(F_n) = 9$. By observation, the Lucas numbers mod 100 have period 60 ; since $60 \mid 7500$, this implies that $L_n \equiv 2 \pmod{100}$. An application of [14, Theorems 6 and 7] and [17] gives that F_n ends in exactly $k + 4$ zeros, and F_{10n} in $k + 5$ zeros. With all this in mind, we write the above formula as

$$\begin{aligned} \left(\frac{F_{10n}}{10^{k+4}} \pmod{100} \right) &= \left(\frac{F_n}{10^{k+4}} \pmod{100} \right) \\ &\quad \cdot [2^9 - 8 \cdot 2^7 + 21 \cdot 2^5 - 20 \cdot 2^3 + 5 \cdot 2] \pmod{100} \\ &= \left(\frac{F_n}{10^{k+4}} \pmod{100} \right) [10] \pmod{100} = 90. \end{aligned}$$

This implies that $\text{lnzd}(F_{10n}) = 9$, as desired. ■

LEMMA 4. For $n = 7500 \cdot 10^k$, $k \geq 0$, and $1 \leq i \leq 9$, then $\text{lnzd}(F_{in}) = 10 - i$.

Proof. The case $i = 1$ is covered by Lemma 3. For $i = 2$ we apply the formula $F_{2n} = F_n L_n$ and the fact (mentioned in Lemma 3) that $\text{lnzd}(L_n) = 2$. The other cases proceed by induction and the formula $F_{in} = F_{(i-1)n} L_n - F_{(i-2)n}$ from [10, p. 92]. ■

LEMMA 5. *For k not a multiple of 5 and for $b \geq 0$, then $\ell(5^b k) = 8^b k \cdot \ell(5^b k - 1)$.*

Proof. $\ell(5^b k) = \text{lnzd}(5^b k!) = \text{lnzd}((5^b k - 1)! \cdot 5^b k)$. It's easy to show that $2^{b+1} | (5^b k - 1)!$, so we can write this last expression as $\text{lnzd}((5^b k - 1)! 2^{-b} \cdot 10^b k) = \text{lnzd}((5^b k - 1)! 2^{-b} \cdot 6^b \cdot 10^b k)$ because 6 acts as the identity on $\{2, 4, 6, 8\}$ under multiplication mod 10. Replacing 6^b with $2^b 8^b$, cancelling the 2's and moving the 8's outside, we get $8^b \cdot \text{lnzd}((5^b k - 1)! k) \pmod{10}$, which gives us $8^b k \cdot \ell(5^b k - 1) \pmod{10}$. ■

LEMMA 6. *For $k \geq 1$ not a multiple of 5 and for $c \geq 1$, then*

$$(1) \ell(5^b k) = 2^b k \cdot \ell(5^b(k - 1)) \pmod{10}.$$

$$(2) \ell(5^b(5c - 1)) = 4 \cdot \ell(5^{b+1}(c - 1)) \pmod{10}.$$

Proof. For $b = 0$, the first statement is trivial and the second follows from applying Lemma 5 four times. We now assume that (1) and (2) hold for $b < a$, and attempt to prove them for $b = a$. For the first,

$$\begin{aligned} \ell(5^a k) &= 8^a k \cdot \ell(5^a k - 1) && \text{(by Lemma 5)} \\ &= 8^a k \cdot 4 \cdot \ell(5 \cdot (5^{a-1} k - 1)) \\ &= 8^a k \cdot 4^2 \cdot \ell(5^2 \cdot (5^{a-2} k - 1)) \\ &\vdots \\ &= 8^a k \cdot 4^a \cdot \ell(5^a \cdot (k - 1)) \\ &= 2^a k \cdot \ell(5^a \cdot (k - 1)) \end{aligned}$$

and for the second, $\ell(5^a(5c-1)) = 2^a(5c-1) \cdot 2^a(5c-2) \cdot 2^a(5c-3) \cdot 2^a(5c-4) \cdot \ell(5^a(5c-5))$ by applying (1) four times, and after simplifying mod 10 we get $4\ell(5^a(5c-5))$, which is $4 \cdot \ell(5^{b+1}(c-1))$. ■

LEMMA 7. For $n = \sum_{i=0}^N a_i 5^i$, then $\ell(n) = 6 \prod_{i=0}^N (a_i!) 2^{i \cdot a_i} \pmod{10}$.

Proof. Suppose 5^{i_0} is the largest power of 5 dividing n . We can thus write $n = 5^{i_0}(m + a_{i_0})$ for m some integer multiple of 5. Applying part (1) of Lemma 6 a_{i_0} times, we get $\ell(n) = 2^{i_0 a_{i_0}} (a_{i_0}!) \ell(5^{i_0} m)$. We now repeat the process with n replaced by $5^{i_0} m = n - a_{i_0} 5^{i_0}$ (and so on) to eventually arrive at the desired formula. ■

For Further Study

For those interested in the history of mathematics, Morris Kline's book [9] is an excellent resource. One also shouldn't ignore Kurt von Fritz's article [15], which gives a delightful discussion of how in the fifth century B.C.E. Hippasus might have discovered irrational numbers by using a pentagon, thus suggesting that the first irrational number could have been the golden ratio, $(1 + \sqrt{5})/2$. Of course, the golden ratio is also the limit of the ratio of successive Fibonacci numbers, which brings us back to one of the subjects of this article.

For those interested in creating more transcendental numbers like the three given in this article, a good place to start might be with other famous sequences such as the squares or the triangular numbers. Euler and Sadek [6] suggest looking at the last digit of the primes p , or perhaps of p^p . If one wishes more sequences to study, there are well over a hundred thousand of them at N. J. A. Sloane's On-Line Encyclopedia of Integer Sequences (<http://www.research.att.com/~njas/sequences/>).

Finally, J. Siehler suggested looking at algebraic numbers. Is there a way to recognize or to perhaps write down a decimal expansion for an algebraic

number, using a theorem similar to the two theorems of this paper? What would such a number look like? It might be easy to prove such a number is algebraic yet hard to actually find its minimal polynomial!

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