On the Mahler Measure of $P(f/g)$.

by Gregory Dresden

Let $P(x)$, $f(x)$, and $g(x)$ be polynomials with integer coefficients. Recently, Rhin and Smyth [6] (using results by Beukers and Zagier [1]) found a lower bound for the Mahler measure of the composition $P(f(x))$ (given certain mild restrictions on P and f) that depends only on f and that grows exponentially with the degree of P. In this paper, we replace $P(f)$ with $P(f/g)$, and establish similar results. (Note: here and in what follows, $P(f/g)$ will refer to the polynomial $P(f(x)/g(x)) \cdot g(x)^{\deg P}$.)

Our main theorem is:

Theorem 1 *For P, f, g integer polynomials such that:*

1. $\deg f > \deg g \geq 0$,

2. *f, g relatively prime,*

3. *either $f(0)$ or $g(0) = 0$,*

4. $\deg P \geq 2$,

5. *If $f(0) = 0$ then $f(x) \neq r g(1/x) \cdot x^{\deg f}$, and if $g(0) = 0$, then $\dfrac{f(x)}{g(x)} \neq r \pm \dfrac{f(1/x)}{g(1/x)}$ (for r rational),*

6. *$P(f/g)$ irreducible,*

then there exists a computable constant c, depending only on f and g, such that

$$M(P(f/g)) \geq c^{\deg P} > 1 \tag{1}$$

For background, see [2], [6], [9]. Briefly, though, the Mahler measure $M(q)$ of an integer polynomial $q(x) = q_* \prod_i (x - \alpha_i)$ is $M(q) = |q_*| \prod_i \max\{1, \log|\alpha_i|\}$. An old theorem of Kronecker tells us that $M(q) = 1$ iff $q(x)$ is a product of cyclotomic polynomials. The smallest known Mahler measure greater than 1 is at about $1.17628\ldots$; however, it is not known if this really is the smallest such value.

We have a number of comments to make before proceeding to the proof.

First, our result is a generalization of (and relies heavily on) the paper [6] by Rhin and Smyth, which in turn generalized earlier works by Zagier [9] and by Zhang [10] on $P(x^2 - x)$. Note that if we take $g(x) = 1$ then our theorem reduces to [6, Theorem 2] (in particular, our conditions 3 and 5, along with $g = 1$, result in $f(x)$ being of degree $t \geq 2$, divisible by x but not by x^t). Note also that [6, Theorem 1] covers the case where $P(f)$ is not necessarily irreducible, and we could probably do the same for $P(f/g)$ if we so desired. However, the lower bound would probably be a bit smaller.

Second, we can view our result, and the results cited above, as establishing non-trivial lower bounds for subsets of $S = \{\frac{1}{\deg P} \log M(P(f/g))\}$. Indeed, a fair amount of attention has been paid to understanding more about this set. For the particular case of $P(x^2 - x)$ alluded to earlier, Zagier [9] found the first few values and Doche [3, 4] found limit points and a region of density for S. For the general case of $P(f/g)$ (with f, g fixed) Dresden [5] found an effectively-computable limit point d such that this set S is dense in $[d, \infty)$ (with mild restrictions on f and g). Similar results hold for $P(x)$ when P is a totally-real polynomial;

see [7, 8].

Third, we point out that our theorem covers more cases than it may initially appear. Suppose, for example, that $\deg f < \deg g$. If we define $Q(u) = P(1/u)\cdot u^{\deg P}$, then $Q(g/f) = P(f/g)$, as shown here (recall that $Q(g/f)$ represents the polynomial $Q(g(x)/f(x))\cdot f(x)^{\deg Q}$):

$$\begin{aligned} Q(g/f) &= Q\left(\frac{g(x)}{f(x)}\right)\cdot f(x)^{\deg Q} \\ &= P\left(\frac{f(x)}{g(x)}\right)\cdot\left(\frac{g(x)}{f(x)}\right)^{\deg P}\cdot f(x)^{\deg Q} \\ &= P\left(\frac{f(x)}{g(x)}\right)\cdot g(x)^{\deg P} \\ &= P(f/g). \end{aligned}$$

Now let us suppose that f and g have the same degree. From the above discussion, we can assume that for f_* and g_* the leading coefficients of $f(x)$ and $g(x)$ respectively, then $|f_*| \geq |g_*|$ (if not, simply replace $P(f/g)$ with $Q(g/f)$). Write $f_* = ng_* + r$, for $0 \leq r < |g_*|$, and let $R(x) = P(x+n)$. Then, $P(f/g) = R((f/g) - n) = R((f-ng)/g) = R(f'/g)$, where the leading coefficient of f' is r (with $|r| < |g_*|$). We can then replace $R(f'/g)$ with $S(g/f')$, and continue this process (which is simply a variant of the Euclidean Algorithm) until the numerator and denominator of our rational polynomial f/g have different degrees. Thus, in conclusion, there is no loss of generaltity in requiring condition 1 in our theorem.

Likewise, it is clearly no restriction to assume (in condition 2) that f and g are relatively prime. However, our condition 3 is indeed restrictive, and it is unclear how to cover the case where $f(0)$ and $g(0)$ are both non-zero; most likely, a new technique will be needed in this situation.

Finally, our restrictions in conditions 4 and 5 serve to eliminate the many cases where we can get a cyclotomic polynomial (thus with Mahler measure 1). While $P(f)$ is

Table 1: $P(f/g)$ **cyclotomic, with** $f(0) = 0$**.**

$P(u)$	$f(x)$	$g(x)$	$P(f/g)$
$u+1$	$\Phi_n(x) - g(x)$	$g(x)$	Φ_n
u^2+u+1	$x^2 - 2x$	$-2x+1$	${\Phi_6}^2$
	$x^4 - x^3 + x$	$x^3 - x + 1$	$\Phi_6\Phi_9$
	$x^4 - 2x$	$-2x^3+1$	$\Phi_3{\Phi_6}^2$
u^2-u+1	x^2+2x	$2x+1$	${\Phi_6}^2$
	$x^4 - x^2 - x$	$-x^3 - x^2 + 1$	Φ_{30}
u^2+1	$x^4 - x^2 - x$	$-x^3 - x^2 + 1$	$\Phi_8\Phi_{12}$
$\Phi_a(u)$	x^t, for $t\|a$	1	Φ_{at}

itself cyclotomic only for $f(x) = x^t$ and $P(x) = \Phi_n(x)$ (the nth cyclotomic polynomial), when we extend the situation to $P(f/g)$, it becomes much more complex. In Table 1, we give a few examples of this in the case $f(0) = 0$. (In the first row, the $g(x)$ can be any polynomial of degree less than $\phi(n)$, such that $\Phi_n(0) - g(0) = 0$.) Note that since $\Phi_a(x^b)$ is either cyclotomic or a product of cyclotomics, we see that we can always generate additional examples by replacing each $f(x), g(x)$ with $f(x^b), g(x^b)$. Turning now to Table 2 (for the case $g(0) = 0$), we see that this trick will again produce additional examples. However, we can also replace f with $f + ag$ (and $P(u)$ with $P(u-a)$) to give a new representation for the composition $P(f/g)$. (In the first row of Table 2, we again require that $g(x)$ be of degree less than $\phi(n)$, but this time such that $g(0) = 0$.) For both tables, note that all these examples will be eliminated by conditions 4 and 5 in the statement of our theorem.

We now proceed with some technical lemmas necessary for our proof.

Lemma 1 *For f, g relatively prime integer polynomials, $t = \deg f > \deg g \geq 0$ and $f(0) = 0$, such that $f(x) - \beta g(x)$ and $[f(1/x) - \beta' g(1/x)]\, x^t$ have the same roots with the same*

Table 2: $P(f/g)$ **cyclotomic, with** $g(0) = 0$**.**

$f(x)$	$g(x)$	$P(u)$	$P(f/g)$
$\Phi_n(x) - g(x)$	$g(x)$	$u + 1$	Φ_n
$x^2 + 1$	x	$u^2 - 2$	Φ_8
		$u^3 - 3u - 1$	Φ_{18}
		$u^4 - 5u^2 + 5$	Φ_{20}
$x^3 - x^2 + 1$	$x^2 - x$	$u^2 + u + 1$	$\Phi_6\Phi_{12}$
		$u^2 + u + 2$	Φ_{14}
		$u^4 + 2u^3 + 4u^2 + 3u + 1$	$\Phi_{10}\Phi_{15}$

multiplicities, and such that β' is irrational, then $f(x) = rg(1/x)x^t$, for r rational.

Proof: We begin by writing:

$$\begin{aligned} f(x) &= f_t x^t + \ldots + f_2 x^2 + f_1 x \\ g(x) &= \qquad g_{t-1} x^{t-1} + \ldots + g_1 x + g_0, \end{aligned}$$

with the understanding that f_t and g_0 are non-zero. We now consider the two polynomials $g_0\beta'\left[f(x) - \beta g(x)\right]$ and $f_t\left[\beta' g(1/x) - f(1/x)\right]x^t$. By comparing coefficients, we see that these two polynomials have the same leading coefficient (of $f_t g_0 \beta'$), and since they have the same roots with the same multiplicities, then these polynomials are actually equal. Thus, equating the constant coefficients gives us $\beta\beta' = (f_t/g_0)^2$, and in equating the ith coefficient we get the following sequence of equations:

$$\begin{aligned} g_0\beta'\left[f_i - \beta g_i\right] &= f_t\left[\beta' g_{t-i} - f_{t-i}\right] \\ g_0\beta' f_i - g_0\beta\beta' g_i &= f_t\beta' g_{t-i} - f_t f_{t-i} \\ \beta'\left(f_i g_0 - f_t g_{t-i}\right) &= \left(g_0 g_i \frac{{f_t}^2}{{g_0}^2} - f_t f_{t-i}\right) \end{aligned}$$

Since β' is irrartional, both terms in parenthesis in the above equation must be zero, and thus we get $f_i g_0 = f_t g_{t-i}$ and $f_t g_i = f_{t-i} g_0$, which leads to $f_i = \frac{f_t}{g_0} g_{t-i}$. As a result, $f(x) = \frac{f_t}{g_0} g(1/x) x^t$, as desired.

We note that we can easily prove the following corollary:

Corollary 1 *With f,g as in Lemma 1, and P an irreducible integer polynomial of degree ≥ 2, such that $P(f/g)$ is a product of cyclotomics, then $f(x) = \pm g(1/x) x^t$.*

Proof: With notation as in the proof of Lemma 1, we see that $P(f/g)$ being a product of cyclotomics implies that f_t and g_0 are ± 1. Take β a root of $P(x)$, so $f(x) - \beta g(x)$ is a factor of $P(f/g)$ and thus has roots ζ_i, all roots of unity. In particular, $f(\zeta_i) - \beta g(\zeta_i) = 0$, but after taking complex conjugates we have $f(1/\zeta_i) - \overline{\beta} g(1/\zeta_i) = 0$. Since this holds for any and all such roots, we can conclude that the conditions of Lemma 1 hold (with $\beta' = \overline{\beta}$), and this gives us our desired result.

The following lemma, similar to Lemma 1, covers the case when $g(0) = 0$.

Lemma 2 *For f, g relatively prime integer polynomials, $t = \deg f > \deg g \geq 0$ and $g(0) = 0$, such that $f(x) - \beta g(x)$ and $[f(1/x) - \beta' g(1/x)]\, x^t$ have the same roots with the same multiplicities, and such that β' is irrational, then $\frac{f(x)}{g(x)} = r \pm \frac{f(1/x)}{g(1/x)}$, for r rational.*

Proof: We begin by writing:

$$\begin{aligned} f(x) &= f_t x^t + \ldots\ldots\ldots\ldots + f_1 x + f_0 \\ g(x) &= \qquad g_{t-1} x^{t-1} + \ldots + g_1 x, \end{aligned}$$

with the understanding that f_t and f_0 are non-zero. We now consider the two polynomials $f_0\,[f(x) - \beta g(x)]$ and $f_t\,[f(1/x) - \beta' g(1/x)]\, x^t$. As in the proof of Lemma 1, by comparing coefficients we see that these two polynomials have the same leading coefficient (of $f_0 f_t$), and since they have the same roots with the same multiplicities, then these polynomials are actually equal. Thus, equating the constant coefficients gives us ${f_0}^2 = {f_t}^2$, so $f_t/f_0 = f_0/f_t = u$, for $u = \pm 1$. Then, in equating the ith coefficients we get $f_i - \beta g_i = u\,[f_{t-i} - \beta' g_{t-i}]$, which becomes $f_i - u f_{t-i} = \beta g_i - \beta' u g_{t-i}$. This implies that β and β' are linearly dependent over the rationals, or in other words, there exist rationals r_1, r_2 such that $\beta = r_1 + r_2 \beta'$. This transforms our equation involving the ith coefficients into $(f_i - u f_{t-i} - r_1 g_i) = \beta' \,(r_2 g_i - u g_{t-i})$. Since β' is irrartional, both terms in parenthesis in this equation must be zero, and thus we get $f_i = u f_{t-i} + r_1 g_i$ and $g_i = (u/r_2) g_{t-i}$. This second equation leads to $g_i = (u/r_2)^2 g_i$, and since $g_i \neq 0$ for some i, we can conclude that $r_2 = \pm 1$ and so $g_i = u r_2 g_{t-i}$. Now, this implies $g(1/x)x^t = g_1 x^{t-1} + \ldots + g_{t-1} x = u r_2 g(x)$. Also, $f(1/x)x^t = u f(x) + r_1 g(1/x) x^t$, which becomes $f(1/x)x^t = u f(x) + r_1 u r_2 g(x)$. Thus,

$$\frac{f(1/x)}{g(1/x)} = \frac{u f(x) + u r_1 r_2 g(x)}{u r_2 g(x)} = r_1 + \frac{1}{r_2}\frac{f(x)}{g(x)} = r_1 \pm \frac{f(x)}{g(x)}$$

as desired.

As before, we have the following corollary:

Corollary 2 *With f,g as in Lemma 2, and P an irreducible integer polynomial of degree ≥ 2, such that $P(f/g)$ is a product of cyclotomics, then* $\dfrac{f(x)}{g(x)} = r \pm \dfrac{f(1/x)}{g(1/x)}$.

The proof is identical to that of the previous corollary, and is hence omitted.

We can now establish the following:

Lemma 3 *Suppose P, f, g integer polynomials, P of degree ≥ 2 with $P(f/g)$ irreducible, f, g relatively prime, $t = \deg f > \deg g \geq 0$, $f(0)g(0) = 0$, such that $f(x) \neq r g(1/x) x^t$ and*

$\dfrac{f(x)}{g(x)} \neq r \pm \dfrac{f(1/x)}{g(1/x)}$ *(for r rational). Then for α_1 a root of $P(f/g)$, there exists α_2 such that $f(\alpha_1)g(\alpha_2) = f(\alpha_2)g(\alpha_1)$ but $f(1/\alpha_1)g(1/\alpha_2) \neq f(1/\alpha_2)g(1/\alpha_1)$.*

Proof: Our proof is an extension and simplification of the proof of [6, Lemma 6]. Let $\beta = f(\alpha_1)/g(\alpha_1)$ and let $\beta' = f(1/\alpha_1)/g(1/\alpha_1)$ So, the equations

$$f(x) - \beta g(x) \qquad \text{and} \qquad [f(1/x) - \beta' g(1/x)]\, x^t \tag{2}$$

share the root α_1. Note that $f - \beta g$, as a factor (over $\mathbf{C}$) of the irreducible $P(f/g)$, has no repeated roots. Also, note that β' is irrational (as if not, then $[f(1/x) - \beta' g(1/x)]\, x^t$ is a minimal polynomial for α_1 over $\mathbf{Q}[x]$, contradicting the fact that $P(f/g)$ irreducible). Suppose, now, that for every α_2 solving the first equation in (2), then α_2 also solves the second. This implies that these equations have the same roots to the same multiplicities (namely, 1). and so we can apply Lemma 1 or Lemma 2 to arrive at the desired contradiction.

Finally, we conclude with:

Proof of Theorem 1: This proceeds exactly as in Rhin and Smyth's paper, [6]. We write: $F(x_{10}, x_{11}, x_{20}, x_{21}) = {x_{10}}^t x_{20}^t \left(f(\frac{x_{11}}{x_{10}}) g(\frac{x_{21}}{x_{20}}) - f(\frac{x_{21}}{x_{20}}) g(\frac{x_{11}}{x_{10}}) \right)$, a bihomogeneous polynomial of degree t. Then,

$$\begin{aligned} F(\alpha_1, 1, \alpha_2, 1) &= {\alpha_1}^t {\alpha_2}^t \left(f(1/\alpha_1) g(1/\alpha_2) - f(1/\alpha_2) g(1/\alpha_1) \right) \\ F(1, \alpha_1, 1, \alpha_2) &= 1 \cdot 1 \cdot \left(f(\alpha_1) g(\alpha_2) - f(\alpha_2) g(\alpha_1) \right), \end{aligned}$$

and by Lemma 3 we can take α_1, α_2 roots of $P(f/g)$ such that $F(1, \alpha_1, 1, \alpha_2) = 0$ but $F(\alpha_1, 1, \alpha_2, 1) \neq 0$. So, the point $(1, \alpha_1, 1, \alpha_2)$ is on $F(\mathbf{x}) = 0$ but not on $F(1/\mathbf{x}) = 0$. Then, we can apply Proposition 4 of [1] (see also [6, Proposition 4]) to conclude that $M(P(f/g)) \geq \rho^{1/4}$, where ρ is the single real root larger than 1 of $1/x^2 + c_F/x^{t-1/2} - 1 = 0$, for $0 < c_F \leq 1$ depending on the coefficients of f and g.

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