A New Approach to Rational Values of Trigonometric Functions

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For a, b both integers, when is $\sin(a\pi/b)$ a rational number? For that matter, what about tan and cos? We all know about the "obvious values" of a and b that will give rational answers:

$$\begin{aligned} \sin(0) = \tan(0) &= 0 \qquad\qquad \cos(0) = 1 \\ \sin(\pi/6) = \cos(\pi/3) &= 1/2 \\ \tan(\pi/4) &= 1 \\ \cos(\pi/2) &= 0 \qquad\qquad \sin(\pi/2) = 1 \end{aligned}$$

...and so on. (For ease of discussion, let's keep a/b in the interval $[0, 1/2]$.)

Are there any <u>other</u> values for a/b such that $\sin(a\pi/b)$ (or cos, or tan) is rational? The answer, of course, is no (as one colleague quipped, if it was rational anywhere else, we surely would have heard about it!). Let us express this fact in the following informal manner:

Fact 1 *For a, b relatively prime integers (with $b > 0$), then $\sin(a\pi/b)$, $\cos(a\pi/b)$, and $\tan(a\pi/b)$ are rational only at the obvious values of a/b (in particular, b can not be other than $1, 2, 3, 4,$ or 6).*

The classical proofs of this fact involve the Chebyshev polynomials and various trig identities (see [1], [3, section 6.3], and [5], as well as the commentary after [4]). Chebyshev polynomials rarely appear in the traditional undergraduate curriculum, and thus the proof of Fact 1 is not usually seen by students. In this paper, we utilize a different procedure, and show that Fact 1 is in fact equivalent to the following well-known statement, familiar to most algebra students:

Fact 2 *For c, d relatively prime integers (with $d > 0$), the primitive dth root of unity $e^{2\pi i c/d}$ has degree ≤ 2 over $\mathbf{Q}$ iff $d = 1, 2, 3, 4,$ or 6.*

We point out that this topic is well suited for an abstract algebra class, and provides a delightful application of the theory of field extensions. The method outlined here is relatively straightforward and would involve only a few minutes of classroom time (alternatively, it would make an excellent homework assignment). Indeed, proving Fact 2 independently takes very little work; one might first show that the degree of the field $\mathbf{Q}(e^{2\pi ic/d})$ is $\phi(d)$ (perhaps by showing that the cyclotomic polynomial $\Phi_d(x)$ is irreducible) and one could then show that $\phi(d) \le 2$ only for the values of d given above. We leave the details as an exercise for the reader (see [2, chapter 33]).

Let us now show the equivalence of our two facts.

First, suppose Fact 1 is true. Let c, d be relatively prime integers (with $d > 0$), and suppose $K = \mathbf{Q}(e^{2\pi ic/d})$ is of degree 2 over $\mathbf{Q}$. By Euler's formula, we can write this primitive dth root of unity as $e^{2\pi ic/d} = \cos(2\pi c/d) + i\sin(2\pi c/d)$. With this in mind, we note that the field K contains the real number $(1/2)\left(e^{2\pi ic/d} + 1/e^{2\pi ic/d}\right) = \cos(2\pi c/d)$ and thus also the complex number $i\sin(2\pi c/d)$. These can't both be degree 2 over $\mathbf{Q}$, as the field K, being only of degree 2, can't contain both a real degree-2 subfield and a complex degree-2 subfield. Thus, either $\sin(2\pi c/d) = 0$ or $\cos(2\pi c/d) \in \mathbf{Q}$. By Fact 1, the first case gives $d = 1$ or 2, and the second gives $d = 1, 2, 3, 4$, or 6.

Second, suppose Fact 2 is true. Choose a rational number $a/b \neq 0$ in reduced form such that $\tan(a\pi/b)$ equals some rational number r, and let $v = 1 + ri$ (see Figure 1, below). Now, v is in $\mathbf{Q}(i)$, but since it's not of length 1, it clearly is not a root of unity and so we can't use Fact 2. So, it would be reasonable to consider

$$\frac{v}{|v|} = \frac{1}{\sqrt{1+r^2}} + \frac{r}{\sqrt{1+r^2}}i = e^{\pi ia/b},$$

which clearly has length 1 and argument $a\pi/b$, and thus is a root of unity. Unfortunately, this complex number is in the possibly degree-4 field $\mathbf{Q}(\sqrt{1+r^2}, i)$ so we still can't apply Fact 2! Instead, we look at

$$\left(\frac{v}{|v|}\right)^2 = \frac{1-r^2}{1+r^2} + \frac{2r}{1+r^2}i = e^{2\pi ia/b},$$

which is clearly in the quadratic number field $\mathbf{Q}(i)$. Thus, by Fact 2 (and since a, b are relatively prime) we have that $b = 1, 2, 3, 4$, or 6; a simple calculation shows that tangent is rational only at the obvious values.

Figure 1

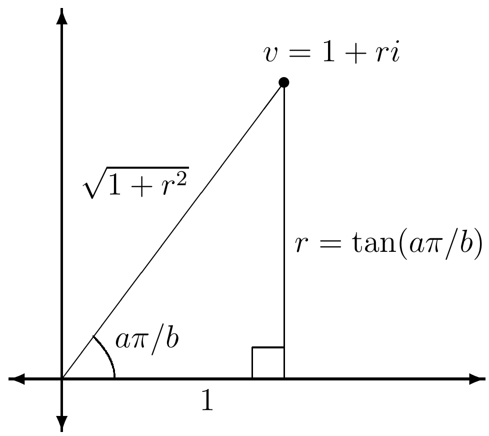
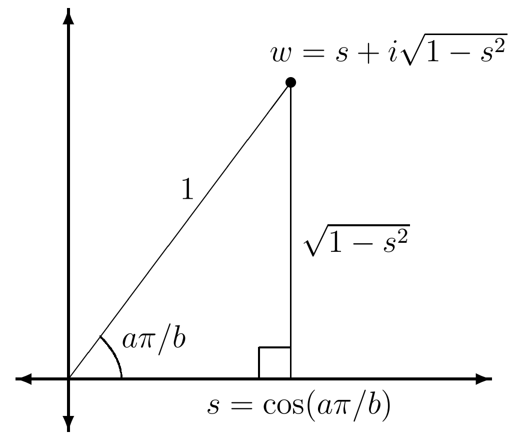


Figure 2



We now proceed to show the same holds for cosine (once we have this, the rationality of sine follows from the identity $\sin(\theta) = \cos(\pi/2 - \theta)$. In a similar manner to our work earlier, we choose a rational number $a/b \neq 0$ in reduced form such that $\cos(a\pi/b) = s$ (for s some rational number), and let $w = s + i\sqrt{1-s^2}$ (see Figure 2, above). Now, $|w| = 1$ and $\arg(w) = a\pi/b$, so $w = e^{ia\pi/b}$ and is in $\mathbf{Q}(i\sqrt{1-s^2})$, a (complex) quadratic number field. Thus, we can apply Fact 2 to note that b must be $1, 2, 3, 4$, or 6, and again, calculations give us the desired obvious values.

This completes our proof of the equivalence of the two facts, but it does not mark the end of this intriguing area of study. For example, we note that the roots of unity of degree 4 are those numbers $e^{2\pi ic/d}$ with $d \in \phi^{-1}(4) = \{5, 8, 10, 12\}$. Likewise, we note that:

$$\cos(\pi/5) = \frac{\sqrt{5}+1}{4} \qquad \sin(\pi/10) = \frac{\sqrt{5}-1}{4}$$
$$\tan(\pi/8) = \sqrt{2} - 1 \qquad \tan(\pi/12) = 2 - \sqrt{3},$$

all simple radicals of degree 2 over $\mathbf{Q}$. The interested reader might want to generalize Facts 1 and 2 to include this correspondence (as well as others of arbitrary degree). Indeed, this might well lead to an alternate proof of the well-known statement that the trig functions are algebraic at all rational multiples of π.

REFERENCES

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