Two Irrational Numbers That Give the Last Non-Zero Digits of $n!$ and n^n.

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Author's Note: This is a slightly revised version of the article that appeared in print in *Mathematics Magazine* in 2001. The original proof of Theorem 2 was incorrect; I've fixed that mistake here. My thanks to Antonio M. Oller-Marcén and José Mara Grau for pointing out the error. Also, a sequel to this paper appeared in 2008.

We begin by looking at the pattern formed from the last (i.e. unit) digit of n^n. Since $1^1 = 1$, $2^2 = 4$, $3^3 = 27$, $4^4 = 256$, and so on, we can easily calculate the first few numbers in our pattern to be $1, 4, 7, 6, 5, 6, 3, 6 \ldots$. We construct a decimal number $N = 0.d_1 d_2 d_3 \ldots d_n \ldots$ such that the n^{th} digit d_n of N is the last (i.e. unit) digit of n^n; that is, $N = 0.14765636\ldots$. In a recent paper [**1**], R. Euler and J. Sadek showed that this N is a rational number with a period of twenty digits:

$$N = 0.\overline{14765636901636567490}.$$

This is a nice result, and we might well wonder if it can be extended. Indeed, Euler and Sadek in [**1**] recommend looking at the last <u>non-zero</u> digit of $n!$ (If we just looked at the last digit of $n!$, we would get a very dull pattern of all 0's, as $n!$ ends in 0 for every $n \geq 5$.)

With this is mind, let's define $\text{lnzd}(A)$ to be the last nonzero digit of the positive integer A; it is easy to see that $\text{lnzd}(A) = A/10^i \bmod 10$, where 10^i is the largest power of 10 that divides A. We wish to investigate not only the pattern formed by $\text{lnzd}(n!)$, but also the pattern formed by $\text{lnzd}(n^n)$. In accordance with [**1**], we define the "factorial" number $F = 0.d_1 d_2 d_3 \ldots d_n \ldots$ to be the infinite decimal such that each digit $d_n = \text{lnzd}(n!)$, and we define the "power" number $P = 0.d_1 d_2 d_3 \ldots d_n \ldots$ to be the infinite decimal such that each digit $d_n = \text{lnzd}(n^n)$, and we ask whether these numbers are rational (i.e. are eventually-repeating decimals) or irrational.

Although the title of this article gives away the secret, we'd like to point out that at first glance, our "factorial" number F exhibits a suprisingly high degree of regularity, and a fascinating pattern occurs. The first few digits of F are easy to calculate:

$$\begin{array}{rcl@{\qquad}rcl@{\qquad}rcl@{\qquad}c}
1! & = & \underline{1} & 5! & = & 1\underline{2}0 & 10! & = & 3628\underline{8}00 & \\
2! & = & \underline{2} & 6! & = & 7\underline{2}0 & 11! & = & 39916\underline{8}00 & \\
3! & = & \underline{6} & 7! & = & 50\underline{4}0 & 12! & = & 479001\underline{6}00 & \cdots \\
4! & = & 2\underline{4} & 8! & = & 403\underline{2}0 & 13! & = & 6227020\underline{8}00 & \\
 & & & 9! & = & 3628\underline{8}0\ldots & 14! & = & 87178291\underline{2}00 &
\end{array}$$

Reading the underlined digits, we have:

$$F = 0.1264\ 22428\ 88682\ldots$$

Continuing along this path, we have (to forty-nine decimal places):

$$F = 0.1264\ 22428\ 88682\ 88682\ 44846\ 44846\ 88682\ 22428\ 22428\ 66264\ldots$$

It is not hard to show that (after the first four digits) F breaks up into five-digit blocks of the form $x\ x\ 2x\ x\ 4x$, where $x \in \{2, 4, 6, 8\}$, and the $2x$ and $4x$ are taken mod 10. Furthermore, if we represent these five-digit blocks by symbols ($\dot{2}$ for 22428, $\dot{4}$ for 44846, $\dot{6}$ for 66264, $\dot{8}$ for 88682, and $\dot{1}$ for the initial four-digit block of 1264), we have:

$$F = 0.\dot{1} \qquad \dot{2} \qquad \dot{8} \qquad \dot{8} \qquad \dot{4} \qquad \dot{4} \qquad \dot{8} \qquad \dot{2} \qquad \dot{2} \qquad \dot{6} \qquad \ldots$$

Grouping these symbols into blocks of five and then performing more calculations (with the aid of `Maple`) give us F to 249 decimal places:

$$F = 0.\dot{1}\dot{2}\dot{8}\dot{8}\dot{4}\ \dot{4}\dot{8}\dot{2}\dot{2}\dot{6}\ \dot{2}\dot{4}\dot{6}\dot{6}\dot{8}\ \dot{4}\dot{8}\dot{2}\dot{2}\dot{6}\ \dot{4}\dot{8}\dot{2}\dot{2}\dot{6}\ \dot{8}\dot{6}\dot{4}\dot{4}\dot{2}\ \dot{2}\dot{4}\dot{6}\dot{6}\dot{8}\ \dot{6}\dot{2}\dot{8}\dot{8}\dot{4}\ \dot{2}\dot{4}\dot{6}\dot{6}\dot{8}\ \dot{2}\dot{4}\dot{6}\dot{6}\dot{8}\ \ldots$$

The reader will notice additional patterns in these blocks of five symbols (twenty-five digits). In fact, such patterns exist for any block of size 5^i. However, a pattern is different from a period, and doesn't imply that our decimal F is rational. Consider the classic example of 0.1 01 001 0001 00001 000001 $\ldots$, which has an obvious pattern but is obviously irrational. It turns out that our decimal F is also irrational, as the following theorem indicates:

THEOREM 1. *Let $F = 0.d_1 d_2 d_3 \ldots d_n \ldots$ be the infinite decimal such that each digit $d_n = \mathrm{lnzd}(n!)$. Then, F is irrational.*

As for our "power" number P, it too might seem to be rational at first glance. P is only slightly different from Euler and Sadek's rational number N, as seen here:

$$\begin{aligned} N &= 0.14765\ 6369\underline{0}\ 16365\ 6749\underline{0}\ 14765\ 6369\underline{0}\ 16365\ 6749\underline{0}\ldots \\ \text{and} \quad P &= 0.14765\ 6369\underline{1}\ 16365\ 6749\underline{6}\ 14765\ 6369\underline{9}\ 16365\ 6749\underline{6}\ldots \end{aligned}$$

(Again, calculations were performed by `Maple`.) Despite this striking similarity between P and N, it turns out that P, like F, is irrational:

THEOREM 2. *Let* $P = 0.d_1 d_2 d_3 \ldots d_n \ldots$ *be the infinite decimal such that each digit* $d_n = \operatorname{lnzd}(n^n)$. *Then,* P *is irrational.*

Before we begin with the (slightly technical) proofs, let us pause and see if we can get a feel for why these two numbers must be irrational. There is no doubt that both F and P are highly "regular", in that both exhibit a lot of repetition. The problem is that there are <u>too many</u> patterns in the digits, acting on different scales. Taking P, for example, we note that there is an obvious pattern (as shown by Euler and Sadek in [**1**]) repeating every 20 digits with 1^1, 2^2, $3^3, \ldots, 9^9$ and $11^{11}, 12^{12}, \ldots, 19^{19}$, but this is broken by a similar pattern for $10^{10}, 20^{20}, \ldots, 90^{90}$ and $110^{110} \ldots 190^{190}$, which repeats every 200 digits. This, in turn, is broken by another pattern repeating every 2000, and so on. A similar behaviour is found for F, but in blocks of 5, 25, 125, and so on, as mentioned above. So, in vague terms, there are always "new patterns" starting up in the digits of P and of F, and this is what makes them irrational.

Are there some simple observations that we can make about P and F which might help us to prove our theorems? To start with, we might notice that every digit of F (except for the first one) is even. Can we prove this? Yes, and without much difficulty:

LEMMA 1. *For* $n \geq 2$, *then* $\operatorname{lnzd}(n!)$ *is in* $\{2, 4, 6, 8\}$.

Proof: The lemma is certainly true for $n = 2, 3, 4$. For $n \geq 5$, we note that the prime factorization of $n!$ contains more 2's than 5's, and thus even after taking out all the 10's in $n!$, the quotient will still be even. To be precise, the number of 5's in $n!$ (and thus the number of trailing zeros in its base-10 representation) is $e_5 = \sum_{i=1}^{\infty} \left[n/5^i \right]$, which is strictly less than the number of 2's, $e_2 = \sum_{i=1}^{\infty} \left[n/2^i \right]$ (here, $[\cdot]$ represents the greatest integer function). Hence, $n!/10^{e_5}$ is an even integer not divisible by 10, and so $\operatorname{lnzd}(n!) = n!/10^{e_5} \bmod 10$, which must be in $\{2, 4, 6, 8\}$. This completes the proof.

Another helpful observation is to note that the lnzd function appears to be multiplicative. For example,

$$\begin{aligned} & \operatorname{lnzd}(12) \cdot \operatorname{lnzd}(53) = 2 \cdot 3 && = 6, \\ \text{and} \quad & \operatorname{lnzd}(12 \cdot 53) = \operatorname{lnzd}(636) && = 6. \end{aligned}$$

However, we note that at times this "rule" fails:

$$\begin{aligned} & \operatorname{lnzd}(15) \cdot \operatorname{lnzd}(22) = 5 \cdot 2 && = 10, \\ \text{yet} \quad & \operatorname{lnzd}(15 \cdot 22) = \operatorname{lnzd}(330) && = 3. \end{aligned}$$

So, we can only prove a limited form of multiplicativity, but it is useful none the less:

LEMMA 2. *Suppose* a, b *are integers such that* $\operatorname{lnzd}(a) \neq 5$, $\operatorname{lnzd}(b) \neq 5$. *Then,* lnzd *is multiplicative; that is,* $\operatorname{lnzd}(a \cdot b) = \operatorname{lnzd}(a) \cdot \operatorname{lnzd}(b)$ *mod* 10.

Proof: Let x' denote the integer x without its trailing zeros; that is, $x' = x/10^i$, where 10^i is the largest power of 10 dividing x. (Note that $\operatorname{lnzd}(x) = x' \bmod 10$.) By hypothesis, a' and b' are both $\neq 0 \bmod 5$, and so $(a \cdot b)' \neq 0 \bmod 5$ and so $(a \cdot b)' = a' \cdot b'$. Thus,

$$\begin{aligned} \operatorname{lnzd}(a \cdot b) &= \operatorname{lnzd}((a \cdot b)') = \operatorname{lnzd}(a' \cdot b') = a' \cdot b' \bmod 10 \\ &= (a' \bmod 10) \cdot (b' \bmod 10) = \operatorname{lnzd}(a') \cdot \operatorname{lnzd}(b') = \operatorname{lnzd}(a) \cdot \operatorname{lnzd}(b). \end{aligned}$$

This completes the proof.

We are now ready to supply the proof of Theorem 1, in which we show that F is irrational. The proof is a little technical, but it relies first on assuming that F has a repeating decimal expansion, then on choosing an appropriate multiple of the period λ_0 and choosing an appropriate digit d, in order to arrive at a contradiction.

Proof of Theorem 1: We argue by contradiction. Suppose F is rational. Then F is eventually periodic; let λ_0 be the period (i.e. for every n sufficiently large, then $d_n = d_{n+\lambda_0}$). Write $\lambda_0 = 5^i \cdot K$ such that $5 \nmid K$ (we acknowledge that K could be 1) and let $\lambda = 2^i \cdot \lambda_0 = 10^i \cdot K$. Then, $\operatorname{lnzd}(\lambda) = \operatorname{lnzd}(K)$, and since $5 \nmid K$, then $10 \nmid K$ and so $\operatorname{lnzd}(K) = K \bmod 10$. Note also that $\operatorname{lnzd}(2\lambda) = 2K \bmod 10$. Choose M sufficiently large so that both of the following are true: $\operatorname{lnzd}(10^M + \lambda) = \operatorname{lnzd}(\lambda)$ (this can easily be done by demanding that $10^M > \lambda$), and for all $n \geq M$, then $d_n = d_{n+\lambda_0}$, which of course would then equal $d_{n+\lambda}$. Finally, let $d = \operatorname{lnzd}((10^M - 1)!)$. By Lemma 1, $d \in \{2, 4, 6, 8\}$, and since $10^M! = (10^M - 1)! \cdot 10^M$, then d also equals $\operatorname{lnzd}(10^M!)$.

Since λ is a multiple of the period λ_0, then if we let $A = 10^M - 1 + \lambda$ and $B = 10^M - 1 + 2\lambda$, then:

$$\begin{aligned} d &= \text{lnzd}((10^M - 1)!) = \text{lnzd}(A!) = \text{lnzd}(B!) \\ \text{and} \quad d &= \text{lnzd}(10^M!) = \text{lnzd}((A+1)!) = \text{lnzd}((B+1)!) \end{aligned}$$

Let's now look at the last two terms in the above equation; it is here we will find our contradiction. Note that since $\text{lnzd}(A!) = d$, then $\text{lnzd}(A!) \neq 5$. Also, since $\text{lnzd}(A+1) = \text{lnzd}(10^M + \lambda) = \text{lnzd}(\lambda) = K \bmod 10$, we know that $\text{lnzd}(A+1) \neq 5$. Thus, we can apply Lemma 2 to $\text{lnzd}(A! \cdot (A+1))$ to get:

$$d = \text{lnzd}((A+1)!) = \text{lnzd}(A!) \cdot \text{lnzd}(A+1) = d \cdot K \bmod 10.$$

Likewise, working with B, we find:

$$d = \text{lnzd}((B+1)!) = \text{lnzd}(B!) \cdot \text{lnzd}(B+1) = d \cdot 2K \bmod 10.$$

Combining these two equations, we get:

$$d = dK \bmod 10 \qquad\qquad d = 2dK \bmod 10,$$

and this becomes $d(1-K) = 0 = d(1-2K) \bmod 10$. Since d is even, this implies that $1 - K = 0 \bmod 5$ and $1 - 2K = 0 \bmod 5$, which is a contradiction. Thus, there can be no period λ_0 and so F is irrational. This completes the proof.

We now turn our attention to the "power" number P derived from the last non-zero digits of n^n. This part was more difficult, but a major step was the discovery that the sequence $\text{lnzd}(100^{100})$, $\text{lnzd}(200^{200})$, $\text{lnzd}(300^{300})\ldots$ was the same as the sequence $\text{lnzd}(100^4)$, $\text{lnzd}(200^4)$, $\text{lnzd}(300^4)\ldots$. This relies not only on the fact that $4|100$ but also on the fact that $a^b = a^{b+4} \bmod 10$ for $b > 0$, as used in the following lemma:

Lemma 3. *Suppose* $100 \mid x$. *Then,* $\text{lnzd}(x^x) = (\text{lnzd}\ x)^4$ *mod* 10.

Proof: As in Lemma 2, let x' denote the integer x without its trailing zeros; that is, $x' = x/10^i$, where 10^i is the largest power of 10 dividing x. Now,

$$\begin{aligned} \text{lnzd}(x^x) &= \text{lnzd}((10^i x')^{10^i x'}) \\ &= \text{lnzd}((10^{i \cdot 10^i x'})(x')^{10^i \cdot x'}) \\ &= \text{lnzd}((x')^{10^i \cdot x'}). \end{aligned}$$

Since $10 \not| x'$, then $10 \not| (x')^{10^i \cdot x'}$, and so:

$$\text{lnzd}(x^x) = (x')^{10^i \cdot x'} \bmod 10.$$

Since $100 \mid x$, then $4 \mid 10^i \cdot x'$, and since $(x')^n = (x')^{n+4} \bmod 10$ for every positive n, we have:

$$\begin{aligned} \text{lnzd}(x^x) &= (x')^4 \bmod 10 \\ &= (\text{lnzd } x)^4 \bmod 10. \end{aligned}$$

This completes the proof.

With Lemma 3 at our disposal, the proof of Theorem 2 is now fairly easy.

Proof of Theorem 2: Again, we argue by contradiction. Suppose P is rational. Let λ_0 be the period, and choose j sufficiently large such that $10^j > 100(\lambda_0 + 1)!$ and such that $\text{lnzd}((10^j + n\lambda_0)^{10^j + n\lambda_0}) = \text{lnzd}((10^j)^{10^j})$ for every positive n. Choosing $n = 100(\lambda_0 + 1)(\lambda_0 - 1)!$, we get:

$$\text{lnzd}((10^j + 100(\lambda_0 + 1)!)^{10^j + 100(\lambda_0 + 1)!}) = \text{lnzd}((10^j)^{10^j}).$$

We reduce the left side of the above equation by Lemma 3 and the right side is obviously 1, so we have:

$$(\text{lnzd}(10^j + 100(\lambda_0 + 1)!))^4 \bmod 10 = 1,$$

but since $10^j > 100(\lambda_0 + 1)!$ and $\text{lnzd}(100(\lambda_0 + 1)!) = \text{lnzd}((\lambda_0 + 1)!)$, we can rewrite the above equation as:

$$(\text{lnzd } (\lambda_0 + 1)!)^4 \bmod 10 = 1.$$

Note that by Lemma 1, the only values of $\text{lnzd}((\lambda_0 + 1)!)$ are $2, 4, 6$, and 8, and raising these to the fourth power mod 10 gives us:

$$6 = 1,$$

which is a contradiction. Thus, P is irrational. This completes the proof.

We close by asking the obvious (and very difficult) question: Are F and P algebraic or transcendental? I suspect the latter, but it is only a hunch, and I hope some curious reader will continue along this interesting line of study.

References

[1] R. Euler and J. Sadek, A number that gives the unit digit of n^n, *Journal of Recreational Mathematics*, 29 (1998) No. 3, pp. 203–4.