FINITE SUBGROUPS OF THE EXTENDED MODULAR GROUP

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ABSTRACT. We show that in the extended modular group $\overline{\Gamma} = \mathrm{PGL}(2, \mathbb{Z})$ there are exactly seven finite subgroups up to conjugacy; three subgroups of size 2, one subgroup each of size 3, 4, and 6, and the trivial subgroup of size 1.

1. Introduction. In a recent article [1], Beauville used Galois cohomology to find all finite subgroups (up to conjugacy) of $\mathrm{PGL}(2, \mathrm{K})$ for certain fields K. In this paper, we use elementary methods to do the same for the group $\overline{\Gamma} = \mathrm{PGL}(2, \mathbb{Z})$, often called the *extended modular group*. Although this result can be derived from earlier work (both Klemm [**4,** Satz 7.9] and Newman [**6,** Chapter IX, §14] classify the finite subgroups of $\mathrm{GL}(2, \mathbb{Z})$, from which we can obtain our result on $\mathrm{PGL}(2, \mathbb{Z})$), we feel it deserves more exposure. Klemm's work was in the context of classifying the wallpaper groups, and Newman used theory from linear algebra. Our paper, in contrast, calls upon two recent results (one on free groups using a theorem of Kurosh, the other on symmetry groups) to do the "heavy lifting", and we combine them to get our main result on $\mathrm{PGL}(2, \mathbb{Z})$ using just basic algebra and direct computation.

We define the group $\overline{\Gamma} = \mathrm{PGL}(2, \mathbb{Z}) = \mathrm{GL}(2, \mathbb{Z})/\{\pm \mathrm{I}\}$ as the set of all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with integer coefficients a, b, c, d and $ad - bc = \pm 1$ with the understanding that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$. One nice feature of this group $\overline{\Gamma}$ is that there is an isomorphism from $\overline{\Gamma}$ to a group of functions (called *linear fractional transforms*) under composition, as

follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \frac{ax+b}{cx+d}$$

Thanks to this isomorphism, we can re-write the product of matrices as a composition of linear fractional transforms, and vice-versa.

In what follows, we will use matrix notation and function notation interchangeably. For example, we will use the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, the matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, and the corresponding function $\frac{1}{x}$ to refer to the same object in $\overline{\Gamma} = \mathrm{PGL}(2,\mathbb{Z})$ as convenient. Because of this correspondence, we can define the "determinant" of the function $(ax+b)/(cx+d)$ to be the determinant $ad - bc$ of the corresponding matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. For more on the extended modular group $\overline{\Gamma}$, see for example [**3, 5, 7, 8**].

2. Statement of Main Result. To find all finite subgroups of $\overline{\Gamma} = \mathrm{PGL}(2,\mathbb{Z})$ up to conjugacy, we need to carefully stitch together two previous results from 2003 and 2004.

This first theorem, by Yılmaz Özgür and Şahin [**9,** Theorem 2.3], comes from considering the presentation of $\overline{\Gamma}$ as a free group with three generators, and it gives us *elements* in $\overline{\Gamma}$ of finite order, up to *conjugacy*.

Theorem 1 (Yılmaz Özgür, Şahin). *There are exactly four conjugacy classes for non-trivial elements of finite order in $\overline{\Gamma}$. Every element of order two is conjugate to either $1/x$ or $-x$ or $-1/x$, and every element of order three is conjugate to $-1/(x+1)$.*

This second result, derived from a paper by Dresden [**2**], comes from considering the finite symmetry groups of the sphere, and it gives us *subgroups* in $\overline{\Gamma}$ of finite order, up to *isomorphism*.

Theorem 2. *There are exactly four isomorphism classes for non-trivial subgroups of finite order in $\overline{\Gamma}$. Every such subgroup is isomorphic to either one of the cyclic groups C_2, C_3, or one of the dihedral groups D_2, D_3, of sizes 2, 3, 4, and 6 respectively.*

(We will prove this theorem in a moment.) We will be able to combine these two theorems to prove our main result, which we state here.

Theorem 3. *Any finite non-trivial subgroup of $\overline{\Gamma} = \mathrm{PGL}(2, \mathbb{Z})$ is of size two, three, four, or six. The groups of size two are conjugate in $\overline{\Gamma}$ to either $\{x, -x\}$ or $\{x, 1/x\}$ or $\{x, -1/x\}$. All groups of size three in $\overline{\Gamma} = \mathrm{PGL}(2, \mathbb{Z})$ are conjugate in $\overline{\Gamma}$ to*

$$G_3 = \left\{ x, \; \frac{-1}{x+1}, \; \frac{-x-1}{x} \right\}.$$

Likewise, all groups of size four are conjugate to

$$G_4 = \left\{ x, \; \frac{1}{x}, \; -x, \; \frac{-1}{x} \right\}.$$

and all groups of size six are conjugate to

$$G_6 = \left\{ x, \; \frac{-1}{x+1}, \; \frac{-x-1}{x}, \; \frac{1}{x}, \; \frac{-x}{x+1}, \; -x-1 \right\}.$$

3. Proofs.

Proof of Theorem 2. Thanks to the isomorphism mentioned earlier, we can think of $\overline{\Gamma} = \mathrm{PGL}(2, \mathbb{Z})$ as the group of linear fractional transforms $(ax + b)/(cx + d)$ with integer coefficients and determinant $ad - bc = \pm 1$. This group $\overline{\Gamma}$ sits inside the larger group of such linear fractional transforms with *non-zero* determinant $ad - bc$, and we call upon Theorem 1 of [**2**] to see that all non-trivial finite subgroups of this larger group (and hence of our group $\mathrm{PGL}(2, \mathbb{Z})$) are isomorphic to either C_2, C_3, C_4, C_6, D_2, D_3, D_4, or D_6. By Theorem 1 of this paper we see that $\mathrm{PGL}(2, \mathbb{Z})$ does not have elements of order 4 or 6, thus eliminating from consideration the groups C_4, C_6, D_4, and D_6. It remains to show that the other finite groups are realizable in $\mathrm{PGL}(2, \mathbb{Z})$, but this follows from the explicit examples given in the statement of our Theorem 3. $\square$

The following proposition is essential to our proof of Theorem 3, and will allow us to combine together our Theorems 1 and 2, above.

Proposition 4. *If a subgroup of $\overline{\Gamma} = \mathrm{PGL}(2, \mathbb{Z})$ contains $-1/x$ and is of size 4, then it must equal G_4. Likewise, if a subgroup of $\overline{\Gamma} = \mathrm{PGL}(2, \mathbb{Z})$ contains $-1/(x+1)$ and is of size 6, then it must equal G_6.*

Proof. We begin by noting that any element in $\mathrm{PGL}(2, \mathbb{Z})$ of order 2 must have matrix form $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$. This is easy to see if we note that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix},$$

and for this to equal the identity in $\mathrm{PGL}(2, \mathbb{Z})$ either we have $d = -a$ as desired, or we have $b = c = 0$ (which forces a^2 and d^2 to be 1, and combined with our matrix being of order 2 and not order 1, this forces $d = -a$ as desired). We also recall that our definition of $\mathrm{PGL}(2, \mathbb{Z})$ requires that $a^2 + bc = \pm 1$.

We turn now to subgroups of $\mathrm{PGL}(2, \mathbb{Z})$ of size 4. Let F_4 be such a subgroup, and suppose F_4 contains $-1/x$. We know from Theorem 2 that F_4 is dihedral, so it contains another element (call it $p(x)$) also of order 2. From our discussion above, we can write $p(x)$ in function form as $p(x) = (ax + b)/(cx - a)$ with $a^2 + bc = \pm 1$, and since a dihedral group of size 4 is abelian then $p(-1/x) = -1/p(x)$. In matrix form, this becomes $\begin{pmatrix} b & -a \\ -a & -c \end{pmatrix} = \pm \begin{pmatrix} c & -a \\ -a & -b \end{pmatrix}$, and by examining the various cases (and recalling that $a^2 + bc = \pm 1$) we conclude that either $p(x) = -x$ or $p(x) = 1/x$, thus giving us F_4 equal to our group G_4.

Finally, we consider a subgroup (call it F_6) of size 6 in $\mathrm{PGL}(2, \mathbb{Z})$ which contains $m(x) = -1/(x+1)$ and thus also $m^2(x) = (-x-1)/x$. From Theorem 2 we know F_6 is dihedral, so it contains another element (call it $p(x)$) also of order 2 such that $m^2(p(x)) = p(m(x))$. As seen earlier, we can write $p(x) = (ax + b)/(cx - a)$ with $a^2 + bc = \pm 1$, and our equality $m^2(p(x)) = p(m(x))$ in matrix form becomes $\begin{pmatrix} -a - c & a - b \\ a & b \end{pmatrix} = \pm \begin{pmatrix} b & b - a \\ -a & -a - c \end{pmatrix}$. If we first consider the "+" in the "$\pm$" above, we quickly arrive at $a = 0$ and thus $b = c = 0$, a contradiction. If we now consider the "$-$" in the "$\pm$" above, we get $b = a + c$, and substituting this into $a^2 + bc = \pm 1$ gives us $a^2 + ac + c^2 = \pm 1$. By looking at the possible values of a and c

(namely, $-1, 0$, and 1) we arrive at $p(x) = -x - 1$ or $p(x) = 1/x$ or $p(x) = -1/(x + 1)$, and so our group F_6 equals G_6 as desired. $\square$

It is now an easy matter to prove our main result.

Proof of Theorem 3. The case for groups of size two and three follows immediately from Theorem 1. If G is a group of size 6, then by Theorem 2 it is dihedral with an element of order three; by Theorem 1 we can conjugate it to get a new group G' containing $-1/(x + 1)$, and by Proposition 4 this new group G' must equal G_6.

The remaining case where G is a group of size four is a bit more challenging. By Theorem 2 we know G is dihedral with three elements of order two; we can thus write $G = \{I, A, B, AB\}$ for A, B, and AB matrices with determinants ± 1. At least one of these three matrices in G must have determinant 1. Now, determinants are preserved under conjugacy in $\mathrm{PGL}(2, \mathbb{Z})$, and by Theorem 1 anything of order 2 in $\mathrm{PGL}(2, \mathbb{Z})$ must be conjugate to either $1/x$ (with associated determinant -1) or $-x$ (with associated determinant -1) or $-1/x$ (with associated determinant 1). Thus, our group G must have an element conjugate to $-1/x$, and so we can conjugate our group G to get a new group G' containing $-1/x$ and then apply Proposition 4 to state that this new group G' must equal G_4. $\square$

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