Chebyshev Polynomials, Sliding Columns, and the k-step Fibonacci Numbers.

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Let's begin by introducing the Chebyshev polynomials of the first kind, defined here as $T_0(x) = 1$, $T_1(x) = x$, and $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$ for $n \geq 2$. To quote from Benjamin and Walton [1],

> *It's hard to avoid Chebyshev polynomials. They appear in just about every branch of mathematics, including geometry, combinatorics, number theory, differential equations, approximation theory, numerical analysis, and statistics.*

We will not go into further detail as to their ubiquity; instead, this article will illustrate an interesting connection between these Chebyshev polynomials and the generalized Fibonacci numbers.

The Fibonacci numbers, of course, are even more well-known than the Chebyshev polynomials; these numbers are defined by $F_n = 0$ for $n \leq 0$, $F_1 = 1$, and thereafter each F_n is the sum of the two previous numbers. A common generalization is to define the k-step Fibonaccis $F_n^{(k)}$ for $k \geq 1$ as follows:

$$F_n^{(k)} = \begin{cases} 0, & \text{if } n \leq 0; \\ 1, & \text{if } n = 1; \\ F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)}, & \text{if } n \geq 2. \end{cases}$$

These are also called the k-generalized Fibonaccis or the k-bonaccis, and for $k = 2$ they give our familiar Fibonaccis: $1, 1, 2, 3, 5, 8, 13, \ldots$. For $k = 3$ they produce what are commonly called the Tribonaccis, and for $k = 4$ the Tetranaccis, and so on.

Returning now to our Chebyshev polynomials, we write out the first seven of them in Table 1. It's interesting to note that the sum of the coefficients along the rising diagonals of Table 1 gives the Fibonacci numbers; Bergum,

$T_0(x)$	$=$	1			
$T_1(x)$	$=$	$1x$			
$T_2(x)$	$=$	$2x^2$	-1		
$T_3(x)$	$=$	$4x^3$	$-3x$		
$T_4(x)$	$=$	$8x^4$	$-8x^2$	$+1$	
$T_5(x)$	$=$	$16x^5$	$-20x^3$	$+5x$	

Table 1: The Chebyshev polynomials $T_n(x)$

Wagner, and Hoggatt proved this in 1975 [2]. As it turns out, if we take the sum of coefficients along rising diagonals of steeper and steeper slopes, we get the Tribonaccis, the Tetranaccis, and so on. Here is the precise statement:

Theorem 1. *For the table of Chebyshev polynomials as given above, and for $k \geq 1$, the sums of coefficients along the lines of slope $k-1$ equal the k-step Fibonaccis.*

A few comments are in order. First, this "sum along rising diagonals" procedure might remind the reader of how the Fibonaccis can also be found by taking sums along the (gently) rising diagonals of Pascal's triangle. This is no coincidence; both the Fibonacci numbers and the Chebyshev polynomials can be written in terms of binomial coefficients. And second, Paolo Serafini in an unpublished note [3] from 2013 used generating functions to come up with the following summation formula for the generalized Fibonacci numbers (with, yes, binomial coefficients),

$$F_n^{(k)} = \sum_{r \leq \lfloor (n-1)/(k+1) \rfloor} (-1)^r 2^{n-2-(k+1)r} \left(\binom{n-1-kr}{r} + \binom{n-2-kr}{r-1} \right),$$

and though Serafini apparently did not realize it at the time, those summands are exactly the coefficients of the $x^{n-1-(k+1)r}$ term in the Chebyshev polynomial $T_{n-1-(k-1)r}$ and so with a bit of effort one can tease out that Serafini's expression is indeed the sum of rising diagonals of slope $k-1$ from Table 1.

Serafini's formula inspired us to find an easier and more natural approach. As a result, our proof uses nothing more than a simple sliding argument and the following key insight: the k-step Fibonaccis also satisfy $F_n^{(k)} = 2F_{n-1}^{(k)} - F_{n-1-k}^{(k)}$ which looks suspiciously like the recurrence relation $T_n = 2xT_{n-1} - T_{n-2}$ for the Chebyshevs.

One issue is that there is only one sequence of Chebyshev polynomials but infinitely many sequences of k-step Fibonacci numbers. Let us now give each k-step Fibonacci sequence $F_n^{(k)}$ its own k-generalized or k-step Chebyshev polynomials, which we call $T_n^{(k)}$ for $k \geq 1$ and which we define as follows:

$$\text{for } k \geq 1,\ \ T_n^{(k)}(x) = \begin{cases} 0, & \text{if } n < 0; \\ 1, & \text{if } n = 0; \\ x, & \text{if } n = 1; \\ 2xT_{n-1}^{(k)}(x) - T_{n-1-k}^{(k)}(x), & \text{if } n \geq 2. \end{cases}$$

Note that when $k = 1$ we get our standard Chebyshev polynomials as defined earlier. For reasons to become clear in a moment, we also need to define 0-step Chebyshevs $T_n^{(0)}$ but with slightly different initial conditions. We set $T_0^{(0)}(x) = 1$, $T_1^{(0)}(x) = x - 1$, and $T_n^{(0)}(x) = 2xT_{n-1}^{(0)}(x) - T_{n-1-0}^{(0)}(x)$ for $n > 1$. That last recurrence relation follows the general pattern of recurrence relations for $T_n^{(k)}$, but in our case of $k = 0$ it simplifies nicely to give us the elegant formula $T_n^{(0)}(x) = (2x-1)^{n-1}(x-1)$. If we write out these polynomials $T_n^{(0)}$ in a table we see a striking similarity to Table 1 earlier; the columns of terms in Table 1

$T_0^{(0)}(x)$	$=$	1						
$T_1^{(0)}(x)$	$=$	$1x$	-1					
$T_2^{(0)}(x)$	$=$	$2x^2$	$-3x$	$+1$				
$T_3^{(0)}(x)$	$=$	$4x^3$	$-8x^2$	$+5x$	-1			
$T_4^{(0)}(x)$	$=$	$8x^4$	$-20x^3$	$+18x^2$	$-7x$	$+1$		
$T_5^{(0)}(x)$	$=$	$16x^5$	$-48x^4$	$+56x^3$	$-32x^2$	$+9x$	-1	

Table 2: The k-step Chebyshev polynomials $T_n^{(k)}(x)$ for $k = 0$

for $T_n(x)$ are exactly the same as the columns in Table 2 for $T_n^{(0)}(x)$, except that each column in Table 1 is pushed down compared to the columns in Table 2. This is always the case, as we describe here:

Lemma 1. *Given any $k \geq 0$, the table of k-step Chebyshev polynomials $T_n^{(k)}(x)$ can be obtained from Table 2 for $T_n^{(0)}(x)$ by sliding down each column of terms in Table 2 so that each column now starts $k+1$ terms below the column to its left.*

In other words, Lemma 1 says that we can form a table for the k-step Chebyshevs $T_n^{(k)}(x)$ by starting with Table 2, leaving the first column alone, sliding the second column down k steps, the third column down $2k$ steps, the fourth column down $3k$ steps, and so on. (Hence, the name *k-step* Chebyshevs.)

Proof of Lemma 1. In what follows, we let k be any non-negative integer. Note that by our recurrence relation $T_n^{(k)}(x) = 2xT_{n-1}^{(k)}(x) - T_{n-1-k}^{(k)}(x)$ and our initial conditions (for both $k = 0$ and $k > 0$), then the leading term of $T_n^{(k)}(x)$ is always

$2^{n-1}x^n$. And again because of the recurrence relation, then each term that's not in the first column is a sum of the term one step above it (times $2x$) and the term $k+1$ steps above it and one step to the left (times -1). But recall that the columns in $T_n^{(k)}(x)$ are all pushed $k+1$ steps down from the columns to their left, and so if we slide those columns back up k terms relative to the previous column, then we are now in the situation where each term in these columns is dependent on the term above it (times $2x$) and the term above and to the left (times -1). But this will give us the table for $T_n^{(0)}(x)$ in which each row is $2x-1$ times the previous row, which means (since the initial conditions are the same) that the terms in the columns are the same for $T_n^{(k)}(x)$ as for $T_n^{(0)}(x)$, as desired. □

We conclude with our final arguments to prove the main theorem.

Proof of Theorem 1. Recall that the k-step Chebyshevs have the recurrence relation $T_n^{(k)}(x) = 2xT_{n-1}^{(k)}(x) - T_{n-1-k}^{(k)}(x)$, which when x is replaced by 1 becomes:

$$T_n^{(k)}(1) = 2T_{n-1}^{(k)}(1) - T_{n-1-k}^{(k)}(1).$$

Now, the k-step Fibonaccis are commonly described as having the recurrence relation $F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)}$, but a simple substitution gives the relation:

$$F_n^{(k)} = 2F_{n-1}^{(k)} - F_{n-1-k}^{(k)}$$

Thus, so long as the $F_n^{(k)}$'s and the $T_n^{(k)}(1)$'s have the same initial conditions (and they do, so long as $k \geq 1$), then they are identical sequences of numbers.

The final step is to note that each number $T_n^{(k)}(1)$ is the sum of the coefficients of $T_n^{(k)}(x)$ along a horizontal line. But by Lemma 1 these coefficients are the same as those for $T_n(x) = T_n^{(1)}(x)$, except that each column is slid up ($k-1$ steps for the second column, $2k-2$ for the third column, and so on) such that what used to be a horizontal line in $T_n^{(k)}(x)$ is now a line of slope $k-1$ in $T_n(x)$, and this is exactly what we need to conclude our proof of Theorem 1. □

(An obvious next step would be to investigate the diagonals of the Chebyshev polynomials of the *second* kind, but that will have to wait for a second paper).

References

[1] Benjamin, A. T., Walton, D. (2009). Counting on Chebyshev polynomials. *Math. Mag.* 82(2): 117–126. doi.org/10.1080/0025570X.2009.11953605

[2] Bergum, G. E., Wagner, W. J., Hoggatt, V. E. (1975). Chebyshev polynomials and related sequences. *Fibonacci Quart.* 13(1): 19–24.

[3] Serafini, P. (2013) A note on the m-step Fibonacci numbers. Preprint.

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