TETRANACCI IDENTITIES VIA HEXAGONAL TILINGS

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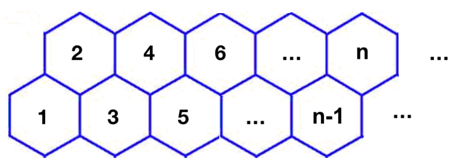
Abstract. We give combinatorial proofs of tetranacci and tetranacci-Fibonacci identities using only squares and dominos on a hexagonal double strip. Some of these are new proofs of old identities, but others we believe have never been seen before.

1. Introduction

As is well known, the Fibonacci numbers count the number of tilings of a *rectangular* $1 \times n$ strip with squares and dominoes. The book *Proofs That Really Count* by Benjamin and Quinn [2] applies this fact to provide numerous combinatorial proofs of Fibonacci identities. Likewise, tetranacci identities can be found by using tiles of lengths 1, 2, 3, and 4, as proved by Benjamin and Heberle [1]. In this paper, we look instead at tiling a *hexagonal* double-strip using only squares and dominos.

First, some definitions. We define an n-strip to be a strip with two rows of a total of n adjacent hexagons, as seen in the figure below. We number the cells starting from the bottom



left corner. In this figure, the six numbered hexagons on the left make up a 6-strip within an n-strip. Note that when tiling this n-strip with squares and dominos, a domino can either be *horizontal* (covering cells k and $k+2$) or *inclined* (covering cells k and $k+1$).

Next, for $n > 0$ let us define T_n to be the number of different ways to tile an n-strip with squares and dominos. Because there is exactly one way to tile a strip of length 0, we define T_0 to equal 1, and for convenience we define $T_n = 0$ for all $n < 0$.

Theorem 1.1. *The sequence T_n is the tetranacci sequence.*

Proof. We can calculate by hand that the first few non-zero values of T_n are $1, 1, 2, 4, 8, 15, 29, \ldots$. We now show that this sequence has the appropriate recurrence relation by conditioning on the last tile of an n-strip (which, by definition, has T_n possible tilings). If this last tile is a square, we remove it to obtain an $(n-1)$-strip which has T_{n-1} tilings. If the last tile is an inclined domino, we again remove it and find T_{n-2} such tilings. If the last tile is a horizontal domino (covering cells n and $n-2$), we condition on the tile covering cell $n-1$; if that tile is a square, there are T_{n-3} ways to tile the remaining $(n-3)$-strip, but if it is another horizontal domino we have T_{n-4} tilings of the remaining strip of length $n-4$. Adding up all such possibilities, we have $T_n = T_{n-1} + T_{n-2} + T_{n-3} + T_{n-4}$ and hence T_n is the tetranacci sequence. $\square$

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2. New Proofs of Old Identities

This first identity was proved algebraically by Howard and Cooper in [3], and combinatorially using tiles of lengths $1, 2, \ldots, r$ on a rectangular board by Benjamin and Heberle in [1]. We use our hexagonal n-strips to provide a third proof,

Identity 2.1. *For $n \neq 0, 1$, we have $2T_{n-1} = T_n + T_{n-5}$.*

Proof. Since $T_n = 0$ for $n < 0$, the identity is easy to check by hand up to $n = 5$. We now assume $n \geq 6$, and (using a technique similar to [2, Identity 7]) we will match all possible tilings of an n-strip and $(n-5)$-strip, with two copies of all tilings of an $(n-1)$-strip, as follows.

For an n-strip ending in a square, we match it with our first copy of an $(n-1)$-strip by simply removing that square. For n-strips ending in an inclined domino, we replace the domino with a square to get an $(n-1)$-strip ending in a square. For n-strips ending in a horizontal domino covering cells n and $n-2$, we need to look at the tile covering cell $n-1$; if that tile is a square, we replace that square on cell $n-1$ and horizontal domino on cells n and $n-2$ with a single inclined domino on cells $n-2$ and $n-1$, giving us an $(n-1)$-strip ending in an inclined domino, but if that tile is another horizontal domino, we replace the first horizontal domino covering cells n and $n-2$ with a square covering just cell $n-2$, giving us an $(n-1)$-strip ending in a horizontal domino and a square.

We have considered all possible n-strips, but there is one $(n-1)$-strip that has not yet been used, and that is the one ending in two stacked horizontal dominos. For this, we simply match it to our $(n-5)$-strip by removing both dominos. □

The formulas in these next two identities were proved algebraically by Waddill in [4], but here we give combinatorial proofs.

Identity 2.2. *For all n, we have $T_{2n} = T_n^2 + T_{n-1}^2 + T_{n-2}^2 + 2T_{n-1}(T_{n-2} + T_{n-3})$.*

Proof. Our technique is similar to [2, Identity 3]. We imagine trying to break a $2n$-strip right in the middle, using a diagonal line that separates cells $n-1$ and n from cells $n+1$ and $n+2$. If there is a clean break, we have two n-strips giving us $(T_n)^2$ tilings. If there is not a clean break, it is due to one of these four possible situations: an inclined domino that covers cells n and $n+1$, two horizontal dominos, a horizontal domino covering cells $n-1$ and $n+1$, or a horizontal domino covering cells n and $n+2$.

For the first case (an inclined domino covering cells n and $n+1$), we simply remove the domino and are left with $(T_{n-1})^2$ ways to tile the two remaining $(n-1)$-strips. For the next case of two horizontal dominos, we do the same and end up with $(T_{n-2})^2$ ways to tile the two remaining $(n-2)$-strips. For the third case (a horizontal domino covering cells $n-1$ and $n+1$), we condition on the tile covering cell n; if this is a square, we have T_{n-2} ways to tile the rest of the left-hand strip and if this is a domino we have T_{n-3}. We multiply this by the T_{n-1} ways to tile the rest of the right-hand strip, and we have in total $T_{n-1}(T_{n-2} + T_{n-3})$ tilings. The fourth case is exactly the same as the third, and by summing the tilings from all the four conditions gives us a total of $T_n^2 + T_{n-1}^2 + T_{n-2}^2 + 2T_{n-1}(T_{n-2} + T_{n-3})$. □

Identity 2.3. *For $n \geq 0$, we have $T_n - 1 = T_{n-2} + 2T_{n-3} + 3(T_{n-4} + T_{n-5} + \cdots + T_1 + T_0)$.*

Proof. Just as in [2, Identity 1], we count how many tilings of an n-strip have at least one domino. On the one hand, there are $T_n - 1$, as we must subtract the one single all-square tiling from the set of all tilings. On the other hand, we condition on the location of the first domino (counting from left to right). If this first domino covers cells k and $k+1$, then to the left of this domino we have just squares, and to to the right there are $T_{n-(k+1)}$ tilings; this applies for k ranging from 1 to $n-1$, giving us $T_0 + T_1 + \cdots + T_{n-2}$ tilings so far. If instead this first domino covers cells k and $k+2$, we must look at the tile covering cell $k+1$. This is either a square, or another domino (covering cells $k+1$ and $k+3$). If a square, then we have $T_{n-(k+2)}$ tilings to the left (with k ranging from 1 to $n-2$), and if a domino, there are $T_{n-(k+3)}$ tilings to the left (with k ranging from 1 to $n-3$). These give us $T_0 + T_1 + \cdots + T_{n-3}$ and $T_0 + T_1 + \cdots + T_{n-4}$ tilings respectively, and when we add everything up we have our desired equation. □

3. New Proofs of New Identities

The following identities appear to be completely new, as we have not found anything similar in the mathematical literature. In what follows, we define a *right-inclined domino* to be a domino which covers cells $2k-1$ and $2k$, and similarly a *left-inclined domino* covers cells $2k$ and $2k+1$.

Lemma 3.1. *There are 2^n ways to tile a $2n$-strip with squares and right-inclined dominos.*

Proof. We tile the $2n$-strip with n right-inclined dominos, and then decide, for each one, whether to keep it or to replace it with two squares. Thus, we have 2^n such tilings. □

Identity 3.2. *For $n \geq 0$, we have* $T_{2n} = 2^n + \sum_{k=1}^{n-1} 2^{n-k}(T_{2k-1} + T_{2k-2} + T_{2k-3}/2)$.

Proof. We condition on the location of the last horizontal or left-inclined domino for a strip of length $2n$. If there is no such domino, then by Lemma 3.1 there are 2^n tilings.

If the last such domino is a left-inclined domino covering cells $2k$ and $2k+1$, there are $2k-1$ open cells to the left which can be tiled T_{2k-1} ways. To the right of this left-inclined domino, cell $2k+2$ must be covered by a square, and that leaves $2n-(2k+2)$ additional open cells to the right which can only be tiled with squares and right-inclined dominos, and by Lemma 3.1 there are 2^{n-k-1} such tilings. Summing up, this give us $\sum_{k=1}^{n-1} 2^{n-k-1} T_{2k-1}$.

If the last such domino is a horizontal domino covering cells $2k$ and $2k+2$, then there are either T_{2k-1} or T_{2k-2} tilings to the left, depending on whether or not cell $2k+1$ is covered by a square, or by a horizontal domino extending back to cell $2k-1$. To the right we again have 2^{n-k-1} tilings, so summing up we have $\sum_{k=1}^{n-1} 2^{n-k-1}(T_{2k-1} + T_{2k-2})$.

Finally, if the last such domino is a horizontal domino covering cells $2k-1$ and $2k+1$, then there are either T_{2k-2} or T_{2k-3} tilings to the left, depending on whether or not cell $2k$ is covered by a square, or by a horizontal domino extending back to cell $2k-2$. To the right we again have 2^{n-k-1} tilings, so summing up we have $\sum_{k=1}^{n-1} 2^{n-k-1}(T_{2k-2} + T_{2k-3})$. When we combine all the terms we obtain our desired formula. □

We now define f_n to be the Fibonacci numbers starting with $f_0 = f_1 = 1$; these count the number of ways to tile a one-dimensional row of n cells with squares and dominos. For convenience, we define $f_{-1} = 0$. To show how these Fibonacci numbers interact with our tetranacci numbers, we begin with two simple identities.

Lemma 3.3. *There are f_n ways to tile an n-strip without horizontal dominos.*

Proof. With no horizontal dominos, any tiling of a n-strip (two rows of adjacent hexagons numbered $1, 2, \ldots, n$) with squares and inclined dominos can be "stretched out" into a one-dimensional row (with cells still numbered $1, 2, \ldots, n$) of squares and dominos. □

Lemma 3.4. *There are f_n ways to tile a $2n$-strip with only dominos.*

Proof. A tiling with only dominos can not have any left-inclined dominos as there would be an odd number of cells to the left of any left-inclined dominos. Hence, all the dominos in any tiling are either horizontal or right-inclined. Consider the bottom row of any such tiling of a $2n$-strip. This bottom row, comprised of n cells labeled $1, 3, 5, \ldots, 2n-1$, is covered with horizontal dominos and the bottom part of right-inclined dominos; this is exactly the same as tiling a one-dimensional row of n cells with dominos and squares. $\square$

Now, we use the two lemmas to prove the following complex identities.

Identity 3.5. *For $n \geq 0$, we have* $T_{2n} = f_n + \displaystyle\sum_{k=1}^{n} T_{2n+1-2k} f_k$.

Proof. We condition on the location of the first square in the tiling of an $2n$-strip. If there is no such square, then by Lemma 3.4 there are f_n such tilings. If the first square is located at the odd cell $2k-1$, then there are only dominos on cells 1 through $2k-2$, and by Lemma 3.4 there are f_{k-1} such tilings; to the right of cell $2k-1$ there are $T_{2n-(2k-1)}$ tilings. Summing up, this gives $\sum_{k=1}^{n} T_{2n+1-2k} f_{k-1}$.

If the first square is located at the even cell $2k$ for $k < n$, then there must be a horizontal domino covering cells $2k-1$ and $2k+1$. This gives f_{k-1} tilings on the left, and $T_{2n-(2k+1)}$ on the right. Summing up, and replacing k with $k-1$, we have $\sum_{k=2}^{n} T_{2n+1-2k} f_{k-2}$. When we combine all the terms, we have our formula. $\square$

Identity 3.6. *For $n \geq 0$, we have* $T_n = f_n + \displaystyle\sum_{k=1}^{n-2} T_{n-2-k} f_k$.

Proof. We condition on the location of the first horizontal domino in the tiling of an n-strip. If there is no such horizontal domino, then by Lemma 3.3 there are f_n such tilings. If the first horizontal domino covers cells k and $k+2$, there are f_{k-1} tilings to the left (again by Lemma 3.3) and either $T_{n-(k+2)}$ or $T_{n-(k+3)}$ tilings to the right, depending on whether cell $k+1$ is covered by a square or by a horizontal domino stretching to cell $k+3$. Summing up we get $\sum_{k=1}^{n-2} T_{n-2-k} f_{k-1}$ and $\sum_{k=1}^{n-3} T_{n-3-k} f_{k-1}$, which we re-index to get $\sum_{k=2}^{n-2} T_{n-2-k} f_{k-2}$. We combine all the terms to obtain our formula. $\square$

Identity 3.7. *For $n \geq 0$, we have* $T_{2n} = f_n^2 + \displaystyle\sum_{k=1}^{n} f_{k-1} \left(f_{k-1} T_{2n-2k} + f_{k-2} T_{2n+1-2k} \right)$.

Proof. We condition on the location of the first inclined domino in the tiling of a $2n$-strip. If there is no such inclined domino, the $2n$-strip can be cut along a horizontal axis into two rows (the bottom with cells $1, 3, 5, \ldots$ and the top with cells $2, 4, 6, \ldots$) each with f_n tilings, giving f_n^2 in all.

If the first inclined domino is right-inclined (covering cells $2k-1$ and $2k$), there are f_{k-1}^2 tilings to the left and T_{2n-2k} to the right. Summing up, we have $\sum_{k=1}^{n} f_{k-1}^2 T_{2n-2k}$.

Finally, if the first inclined domino is left-inclined (covering cells $2k$ and $2k+1$), there are $f_k f_{k-1}$ tilings on the left, and $T_{2n-(2k+1)}$ on the right. Summing up and changing the index gives us $\sum_{k=2}^{n} f_{k-1} f_{k-2} T_{2n+1-2k}$. We combine all the terms to obtain our formula. $\square$

Identity 3.8. *For $n \geq 0$, we have* $T_{2n+1} = f_{n+1} f_n + \displaystyle\sum_{k=1}^{n} f_{k-1} \left(f_k T_{2n-2k} + f_{k-1} T_{2n+1-2k} \right)$.

Proof. Same as the previous identity, replacing the $2n$-strip with a $(2n+1)$-strip. $\square$

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