SUMS AND CONVOLUTIONS OF *K*-BONACCI AND *K*-LUCAS NUMBERS

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Received: 1/16/20, Revised: 1/22/21, Accepted: 5/11/21, Published: 5/17/21

Abstract

We state and prove a variety of formulas for the sums, weighted sums, and convolutions of generalized Fibonacci and Lucas numbers.

1. Introduction

The famous Fibonacci and Lucas numbers need no introduction. Here are two well-known (and strikingly similar) formulas involving the sums of these numbers:

$$\sum_{i=0}^{n} F_i = F_{n+2} - 1, \qquad \sum_{i=0}^{n} L_i = L_{n+2} - 1. \tag{1}$$

One way to generalize the Fibonacci numbers is to define the k-bonacci numbers $F_n^{(k)}$ as $F_n^{(k)} = 0$ for $-(k-2) \le n \le 0$, $F_1^{(k)} = 1$, and then each subsequent $F_n^{(k)}$ as the sum of the k previous terms. For $k = 3$ this gives us the so-called Tribonaccis, and likewise higher values of k give us the Tetranaccis, Pentanaccis, and so on. See Table 1.

k	name	symbol	first few terms starting at $n = 0$
2	Fibonacci	F_n	$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$
3	Tribonacci	T_n	$0, 1, 1, 2, 4, 7, 13, 24, 44, 81, \ldots$
4	Tetranacci	$F_n^{(4)}$	$0, 1, 1, 2, 4, 8, 15, 29, 56, 108, \ldots$
5	Pentanacci	$F_n^{(5)}$	$0, 1, 1, 2, 4, 8, 16, 31, 61, 120, \ldots$

Table 1: The k-bonacci numbers $F_n^{(k)}$ for $k = 2, 3, 4, 5$.

As for generalizing the Lucas numbers, we follow the approach of Noe and Post [9], building on earlier work by Fielder [3]. We set $L_n^{(k)} = -1$ for $-(k-1) \le n \le -1$, we define $L_0^{(k)} = k$, and then we define each subsequent $L_n^{(k)}$ as the sum of the k previous terms. A bit of calculation gives us Table 2.

k	name	symbol	first few terms starting at $n = 0$
2	Lucas	L_n	$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, \ldots$
3	3-Lucas	$L_n^{(3)}$	$3, 1, 3, 7, 11, 21, 39, 71, 131, 241, \ldots$
4	4-Lucas	$L_n^{(4)}$	$4, 1, 3, 7, 15, 26, 51, 99, 191, 367, \ldots$
5	5-Lucas	$L_n^{(5)}$	$5, 1, 3, 7, 15, 31, 57, 113, 223, 439, \ldots$

Table 2: The k-Lucas numbers $L_n^{(k)}$ for $k = 2, 3, 4, 5$.

With this in mind, let us explore the sums of these k-bonacci and k-Lucas numbers. For $k = 3$, the Tribonacci sum formula

$$\sum_{i=0}^{n} T_i = \frac{1}{2}\left(T_{n+2} + T_n - 1\right)$$

was proved by Kiliç [6, Theorem 2] using matrix methods, and the 3-Lucas sum formula

$$\sum_{i=0}^{n} L_i^{(3)} = \frac{1}{2}\left(L_{n+2}^{(3)} + L_n^{(3)} + 0\right)$$

comes from Frontczak's recent article [4]. Notice the similarity to the formulas in Equation (1).

Likewise, for $k = 4$ we have the Tetranacci formula

$$\sum_{i=0}^{n} F_i^{(4)} = \frac{1}{3}\left(F_{n+2}^{(4)} + 2F_n^{(4)} + F_{n-1}^{(4)} - 1\right)$$

from Kiliç [5], and the 4-Lucas formula

$$\sum_{i=0}^{n} L_i^{(4)} = \frac{1}{3}\left(L_{n+2}^{(4)} + 2L_n^{(4)} + L_{n-1}^{(4)} + 2\right)$$

from Soykan [12].

Can we find a single formula for these sums that holds for all possible k? And can we find other formulas for convolutions or weighted sums? The answer to both questions is yes. In addition to generalizing the sums given above, we will also give

k-bonacci and k-Lucas versions of the following well-known formulas:

$$\begin{aligned} \sum_{i=1}^{n} iF_i &= nF_{n+2} - F_{n+3} + 2 \qquad \text{(weighted sums)},\\ \sum_{i=1}^{n-1} F_i L_{n-i} &= (n-1)F_n \qquad \text{(convolutions)},\\ \sum_{i=0}^{n} 2^{n-i} F_i &= 2^{n+1} - F_{n+3} \qquad \text{(falling powers of 2)},\\ \text{and} \quad \sum_{i=0}^{n} 2^i L_i &= 2^{n+1} F_{n+1} \qquad \text{(rising powers of two)}. \end{aligned}$$

In each case, we will present a single formula that holds for all values of k. Aside from Theorem 1 below (on the sum of the k-bonaccis), all our results are new.

Before we begin, let us review a few basic facts about the k-bonacci and k-Lucas numbers. As defined above, the k-bonaccis begin with $F_0^{(k)} = 0$ and $F_1^{(k)} = 1$ and from our definition it is easy to show that for $2 \leq n \leq k+1$ we have $F_n^{(k)} = 2^{n-2}$. Likewise, it is easy to show from our definition of the k-Lucas numbers that for $1 \leq n \leq k$ then $L_n^{(k)} = 2^n - 1$. From Fielder's article [3] we learn that the k-Lucas numbers can also be defined in terms of a Binet-style formula: $L_n^{(k)} = \sum_{i=1}^{k} \alpha_i^n$, where $\alpha_1, \ldots, \alpha_k$ are the k distinct roots of $x^k - (x^{k-1} + \cdots + x + 1)$ and where $n \geq -(k-1)$. This is a natural extension of the Binet formula for the original Lucas numbers, which states that $L_n = \alpha^n + \beta^n$ for α, β the two roots of $x^2 - x - 1$.

We know that $F_n^{(k)}$ and $L_n^{(k)}$ are each the sum of the k previous terms; from this recurrence relation we can easily obtain the equations

$$F_n^{(k)} = 2F_{n-1}^{(k)} - F_{n-k-1}^{(k)} \qquad \text{for all } n \geq 3, \tag{2}$$

$$L_n^{(k)} = 2L_{n-1}^{(k)} - L_{n-k-1}^{(k)} \qquad \text{for all } n \geq 2. \tag{3}$$

Finally, Noe and Post [9] provide this useful relationship between the k-bonacci and the k-Lucas numbers which follows immediately from their generating functions.

$$\text{For } k \geq 2 \text{ and } n \geq 1, \qquad L_n^{(k)} = 1F_n^{(k)} + 2F_{n-1}^{(k)} + 3F_{n-2}^{(k)} + \cdots + kF_{n-k+1}^{(k)}. \tag{4}$$

From this we obtain the following lovely formulas:

$$L_{n+1}^{(k)} - L_n^{(k)} = (k+1)F_{n+2}^{(k)} - 2kF_{n+1}^{(k)}, \tag{5}$$

$$L_{n+1}^{(k)} - L_n^{(k)} = F_{n+2}^{(k)} - kF_{n+1-k}^{(k)}, \tag{6}$$

and

$$1F_{n+1}^{(k)} + 2F_{n+2}^{(k)} + 3F_{n+3}^{(k)} + \cdots + kF_{n+k}^{(k)} = (k+1)F_{n+k+1}^{(k)} - L_{n+k}^{(k)}. \tag{7}$$

Finally, we have this interesting identity:

$$L_{n+k}^{(k)} - L_n^{(k)} = 2F_{n+k+1}^{(k)} - (k+1)F_{n+1}^{(k)}. \tag{8}$$

To show this, we first need to expand the left side of Equation (8) into a telescoping sum and then apply Equation (6) to simplify each term in the sum, which gives us

$$L_{n+k}^{(k)} - L_n^{(k)} = \sum_{i=1}^{k} \left(L_{n+i}^{(k)} - L_{n+i-1}^{(k)} \right) = \sum_{i=1}^{k} \left(F_{n+i+1}^{(k)} - k \cdot F_{n+i-k}^{(k)} \right). \tag{9}$$

By our recurrence relation, the right-hand side of (9) adds up to $F_{n+k+2}^{(k)} - kF_{n+1}^{(k)}$, and if we use (2) to replace that $F_{n+k+2}^{(k)}$ with $2F_{n+k+1}^{(k)} - F_{n+1}^{(k)}$, we obtain $2F_{n+k+1}^{(k)} - (k+1)F_{n+1}^{(k)}$ which gives us the right-hand side of our desired Equation (8).

2. Sum Formulas

Back in 2008, Kiliç [5, Theorem 2] used matrices to prove a version of this first theorem. Ten years later, Schumacher [10, Theorem 4.1] used induction to do the same.

Theorem 1 (Kiliç, Schumacher)**.** *For $k \geq 2$ and $n \geq 0$, we have*

$$\sum_{i=0}^{n} F_i^{(k)} \; = \; \frac{1}{k-1} \left(-1 + \sum_{j=1}^{k} jF_{n-k+j}^{(k)} \right).$$

Thanks to Equation (4), this theorem is equivalent to the following much easier version, which has a surprisingly simple induction proof; we do in a few lines what took Kiliç and Schumacher each a few pages to prove.

Theorem 2. *For $k \geq 2$ and $n \geq 0$, we have*

$$\sum_{i=0}^{n} F_i^{(k)} \; = \; \frac{1}{k-1} \left((k+1)F_{n+1}^{(k)} - L_n^{(k)} - 1 \right).$$

Proof. For $n = 0$, the equation in the theorem clearly holds because $F_0^{(k)} = 0$, $F_1^{(k)} = 1$, and $L_0^{(k)} = k$. If we now assume that the equation holds true for n, we simply add $F_{n+1}^{(k)}$ to both sides. On the left, the $F_{n+1}^{(k)}$ is absorbed into the sum. On the right, we absorb the $F_{n+1}^{(k)}$ into the numerator, which now looks like

$$\left((k+1)F_{n+1}^{(k)} - L_n^{(k)} - 1 + (k-1)F_{n+1}^{(k)} \right) = \left(2kF_{n+1}^{(k)} - L_n^{(k)} - 1 \right),$$

and thanks to Equation (5) this equals $\left((k+1)F_{n+2}^{(k)} - L_{n+1}^{(k)} - 1 \right)$ which completes our induction. □

We note that for $k = 2, 3$ and 4 then this theorem can be used to give us the formulas for the k-bonacci sums on the first page. As for the k-Lucas sums on that same page, we have a new theorem which is quite similar to Theorem 1.

Theorem 3. *For $k \geq 2$ and $n \geq 0$, we have*

$$\sum_{i=0}^{n} L_i^{(k)} = \frac{1}{k-1}\left(k(k-3)/2 + \sum_{j=1}^{k} jL_{n-k+j}^{(k)}\right), \tag{10}$$

or equivalently,

$$\sum_{i=0}^{n} L_i^{(k)} = \frac{1}{k-1}\left(k(k-3)/2 + \sum_{j=1}^{k} jL_{n+j}^{(k)}\right) - L_{n+k+1}^{(k)}. \tag{11}$$

It is possible to prove this by Kiliç's matrix methods or by Schumacher's induction technique, but for variety we will give a direct proof using Fielder's Binet-style definition of the k-Lucas numbers as mentioned earlier.

Proof. For α a root of $x^k - \left(x^{k-1} + x^{k-2} + \cdots + x + 1\right) = 0$, note that

$$(\alpha - 1)\left(1 + 2\alpha + 3\alpha^2 + \cdots + k\alpha^{k-1}\right) = -1 - \alpha - \alpha^2 - \cdots - \alpha^{k-1} + k\alpha^k,$$

which becomes $-\alpha^k + k\alpha^k$ which equals $(k-1)\alpha^k$. Thus, we conclude that

$$\frac{\alpha^k}{\alpha - 1} = \frac{1}{k-1}\left(1 + 2\alpha + 3\alpha^2 + \cdots + k\alpha^{k-1}\right), \tag{12}$$

and after multiplying both sides by α^{n+1-k}, we have

$$\frac{\alpha^{n+1}}{\alpha - 1} = \frac{1}{k-1}\left(\alpha^{n-k+1} + 2\alpha^{n-k+2} + 3\alpha^{n-k+3} + \cdots + k\alpha^n\right). \tag{13}$$

Next, we consider the geometric sum

$$\sum_{i=0}^{n} \alpha^i = \frac{\alpha^{n+1} - 1}{\alpha - 1} = \frac{\alpha^{n+1} - \alpha^k}{\alpha - 1} + \frac{\alpha^k - 1}{\alpha - 1}. \tag{14}$$

The last term on the right of (14) can be rewritten as the geometric sum $1 + \alpha + \cdots + \alpha^{k-1}$ which equals a^k. If we use this, along with Equations (12) and (13), and sum (14) over all the roots $\alpha_1, \ldots, \alpha_k$, we obtain

$$\begin{aligned}\sum_{i=0}^{n} L_i^{(k)} &= \frac{1}{k-1}\left(L_{n-k+1}^{(k)} + 2L_{n-k+2}^{(k)} + 3L_{n-k+3}^{(k)} + \cdots + kL_n^{(k)}\right) \\ &\quad - \frac{1}{k-1}\left(L_0^{(k)} + 2L_1^{(k)} + 3L_2^{(k)} + \cdots + kL_{k-1}^{(k)}\right) + L_k^{(k)}.\end{aligned}$$

We now use that $L_0^{(k)} = k$ and that $L_i^{(k)} = 2^i - 1$ for $1 \le i \le k$ to obtain

$$\begin{aligned}\sum_{i=0}^{n} L_i^{(k)} &= \frac{1}{k-1}\left(\sum_{j=1}^{k} jL_{n-k+j}^{(k)}\right)\\ &\quad - \frac{1}{k-1}\left(k + 2(2^1-1) + 3(2^2-1) + \cdots + k(2^{k-1}-1)\right) \;+\; 2^k - 1.\end{aligned}$$

We apply some basic summation formulas to the second line of the above equation to get

$$\sum_{i=0}^{n} L_i^{(k)} = \frac{1}{k-1}\left(\sum_{j=1}^{k} jL_{n-k+j}^{(k)} - \left((k-1)2^k + 1 - k(k+1)/2\right)\right) \;+\; 2^k - 1,$$

which reduces to our desired formula,

$$\sum_{i=0}^{n} L_i^{(k)} = \frac{1}{k-1}\left(\sum_{j=1}^{k} jL_{n-k+j}^{(k)} \;+\; k(k-3)/2\right). \tag{15}$$

For the second part of the theorem, if we take each $L_{n-k+j}^{(k)}$ in the summation on the right of Equation (15) and replace it with $2L_{n+j}^{(k)} - L_{n+j+1}^{(k)}$ we have

$$\sum_{j=1}^{k} jL_{n-k+j}^{(k)} = \sum_{j=1}^{k} j(2L_{n+j}^{(k)} - L_{n+j+1}^{(k)}).$$

If we break apart the sum on the right and re-index the second part, we find that

$$\sum_{j=1}^{k} jL_{n-k+j}^{(k)} = \sum_{j=1}^{k} 2jL_{n+j}^{(k)} - \sum_{j=2}^{k+1}(j-1)L_{n+j}^{(k)},$$

and re-arranging the terms on the right gives us

$$\sum_{j=1}^{k} jL_{n-k+j}^{(k)} = \sum_{j=1}^{k} jL_{n+j}^{(k)} + \sum_{j=1}^{k} L_{n+j}^{(k)} - k \cdot L_{n+k+1}^{(k)}.$$

The middle sum equals $L_{n+k+1}^{(k)}$, which means we have

$$\sum_{j=1}^{k} jL_{n-k+j}^{(k)} = \sum_{j=1}^{k} jL_{n+j}^{(k)} - (k-1) \cdot L_{n+k+1}^{(k)},$$

and if we substitute this into Equation (15) then we establish Equation (10). $\square$

3. Weighted Sum Formulas

Here are two remarkable formulas for weighted sums.

Theorem 4. *For $k \geq 2$ and $n \geq 0$, we have*

$$\sum_{i=0}^{n} iF_i^{(k)} = \frac{1}{(k-1)^2}\left[1 + k(k-1)/2 \ + \ n(k-1)\sum_{j=0}^{k-1}(k-j)F_{n-j}^{(k)} \ + \ \sum_{j=1}^{k} c_j F_{n+j}^{(k)}\right],$$

and likewise

$$\sum_{i=0}^{n} iL_i^{(k)} = \frac{1}{(k-1)^2}\left[k(k+1) \ + \ n(k-1)\sum_{j=0}^{k-1}(k-j)L_{n-j}^{(k)} \ + \ \sum_{j=1}^{k} c_j (L_{n+j}^{(k)}+1)\right],$$

where $c_j = \big((k-1)j^2 - ((k-1)^2+2)j\big)/2$.

Note that for $k = 2$ the above formulas become $\sum_{i=1}^{n} iF_i = nF_{n+2} - F_{n+3} + 2$ and $\sum_{i=1}^{n} iL_i = nL_{n+2} - L_{n+3} + 4$, which were discussed by Koshy [7, §25.1]. For $k = 3$, the first formula in Theorem 4 gives us the weighted Tribonacci sum formula

$$\sum_{i=0}^{n} iT_i = \frac{1}{2}\Big[2 \ + \ n(T_n + T_{n+2}) \ - \ (T_{n+1} + T_{n+2})\Big],$$

which was discovered independently by Adegoke [1] and Schumacher [11]. Finally, for $k = 4$ we can recreate Schumacher's formula [10, §5] for the weighted sum of the Tetranaccis. Of course, our theorem will also give us the weighted sums of the Pentanaccis, Sextanaccis, Heptanaccis, and so on.

Proof of Theorem 4. We will prove the k-bonacci formula, and leave the proof of the k-Lucas formula as an exercise for the reader. With a bit of work and with extensive use of Equations (5) and (7), the k-bonacci part of Theorem 4 is equivalent to the equation

$$\begin{aligned}\sum_{i=0}^{n} iF_i^{(k)} = \frac{1}{(k-1)^2}\Bigg[&1 + \frac{k(k-1)}{2} \ + \ n(k-1)(2kF_n^{(k)} - L_{n-1}^{(k)}) \\ &+ \ \frac{(k-1)}{2}\sum_{j=1}^{k} j^2 F_{n+j}^{(k)} \ - \ \frac{(k-1)^2+2}{2}\Big((k+1)F_{n+k+1}^{(k)} - L_{n+k}^{(k)}\Big)\Bigg]. \quad (16)\end{aligned}$$

For the base case of $n = 0$, Equation (16) reduces to

$$0 = \frac{1}{(k-1)^2}\Bigg[1 + \frac{k(k-1)}{2} \;+\; \frac{1}{2}(k-1)\sum_{j=1}^{k} j^2 F_j^{(k)} \\ -\; \frac{(k-1)^2+2}{2}\left((k+1)F_{k+1}^{(k)} - L_k^{(k)}\right)\Bigg], \quad (17)$$

and since $F_1^{(k)} = 1$, $F_j^{(k)} = 2^{j-2}$ for $2 \le j \le k-1$, and $L_k^{(k)} = 2^k - 1$, then this becomes

$$0 = \frac{1}{(k-1)^2}\Bigg[1 + \frac{k(k-1)}{2} \;+\; \frac{1}{2}(k-1)\left(1 + \sum_{j=2}^{k} j^2 2^{j-2}\right) \\ -\; \frac{(k-1)^2+2}{2}\left((k+1)2^{k-1} - (2^k - 1)\right)\Bigg]. \quad (18)$$

A bit of arithmetic shows that the right-hand side of Equation (18) does indeed simplify to zero.

For the induction step, we will look at the difference between Equation (16) at $n+1$ and at n; on the left, this difference will be simply $(n+1)F_{n+1}^{(k)}$ and so if we can show that the difference on the right is the same then we have verified the induction. A key step will be to analyze the expression

$$\sum_{j=1}^{k} j^2 F_{n+1+j}^{(k)} \;-\; \sum_{j=1}^{k} j^2 F_{n+j}^{(k)}. \quad (19)$$

If we take (19) and re-index the first sum then we have

$$\sum_{j=2}^{k+1} (j-1)^2 F_{n+j}^{(k)} \;-\; \sum_{j=1}^{k} j^2 F_{n+j}^{(k)},$$

and this simplifies nicely to give us

$$\sum_{j=1}^{k} -2jF_{n+j}^{(k)} \;+\; \sum_{j=1}^{k} F_{n+j}^{(k)} \;+\; k^2 F_{n+k+1}^{(k)}. \quad (20)$$

We use Equation (7) on the first sum in (20), and the recurrence relation for the second sum in (20), to obtain the following nice formula for (19):

$$\sum_{j=1}^{k} j^2 F_{n+1+j}^{(k)} - \sum_{j=1}^{k} j^2 F_{n+j}^{(k)} = -2(k+1)F_{n+k+1}^{(k)} + 2L_{n+k}^{(k)} + (k^2+1)F_{n+k+1}^{(k)} \quad (21)$$

$$= 2L_{n+k}^{(k)} + (k^2 - 2k - 1)F_{n+k+1}^{(k)}. \quad (22)$$

With this, we can now write down the difference between the right-hand side of Equation (16) at $n+1$ and at n. This difference can be grouped into the following three expressions. The first expression is

$$\frac{1}{(k-1)^2}\left[(n+1)(k-1)(2kF_{n+1}^{(k)}-L_n^{(k)}) \;-\; n(k-1)(2kF_n^{(k)}-L_{n-1}^{(k)})\right], \tag{23}$$

the second is

$$\frac{1}{(k-1)^2}\left[\frac{k-1}{2}\sum_{j=1}^{k} j^2 F_{n+1+j}^{(k)} \;-\; \frac{k-1}{2}\sum_{j=1}^{k} j^2 F_{n+j}^{(k)}\right], \tag{24}$$

and the third expression simplifies to

$$\frac{-(k^2-2k+3)}{2(k-1)^2}\left[(k+1)(F_{n+k+2}^{(k)}-F_{n+k+1}^{(k)}) \;-\; (L_{n+k+1}^{(k)}-L_{n+k}^{(k)})\right]. \tag{25}$$

Inside the brackets of our first expression (23), we can simplify this to

$$n(k-1)(2kF_{n+1}^{(k)}-2kF_n^{(k)})-(L_n^{(k)}-L_{n-1}^{(k)})) \;+\; (k-1)(2kF_{n+1}^{(k)}-L_n^{(k)}).$$

We use Equation (5) to replace that $(L_n^{(k)}-L_{n-1}^{(k)})$ with $(k+1)F_{n+1}^{(k)}-2kF_n^{(k)}$, and so the above simplifies nicely to

$$(k-1)\left(2kF_{n+1}^{(k)}-L_n^{(k)}\right)+n(k-1)^2F_{n+1}^{(k)},$$

which means our expression (23) is now

$$\frac{1}{k-1}\left[2kF_{n+1}^{(k)}-L_n^{(k)}\right] \;+\; nF_{n+1}^{(k)}. \tag{26}$$

Turning now to our second expression (24), if we factor out the $k-1$ and apply Equation (22), we obtain

$$\frac{1}{k-1}\left[L_{n+k}^{(k)} \;+\; \frac{(k^2-2k-1)}{2}F_{n+k+1}^{(k)}\right]. \tag{27}$$

As for our third expression (25), we use Equation (5) to replace that last $(L_{n+k+1}^{(k)}-L_{n+k}^{(k)})$ with $(k+1)F_{n+k+2}^{(k)}-2kF_{n+k+1}^{(k)}$, and so (25) simplifies nicely to

$$\frac{1}{k-1}\left[\frac{-(k^2-2k+3)}{2}F_{n+k+1}^{(k)}\right]. \tag{28}$$

We now add together our three expressions (26), (27), and (28) to obtain

$$\frac{1}{k-1}\left[2kF_{n+1}^{(k)}-L_n^{(k)}+L_{n+k}^{(k)}-2F_{n+k+1}^{(k)}\right] \;+\; nF_{n+1}^{(k)}, \tag{29}$$

and if we apply Equation (8) we obtain

$$\frac{1}{k-1}\left[2kF_{n+1}^{(k)}-(k+1)F_{n+1}^{(k)}\right]\ +\ nF_{n+1}^{(k)}\ \ =\ \ (n+1)F_{n+1}^{(k)}. \tag{30}$$

As mentioned earlier, this $(n+1)F_{n+1}^{(k)}$ is the difference between the left-hand side of Equation (16) at $n+1$ and at n, and so we are done. $\square$

4. Convolutions

Two well-known convolution formulas [2, 14] involving the traditional Fibonacci and Lucas numbers can be written as

$$\sum_{i=1}^{n-1} F_i L_{n-i} = (n-1)F_n$$

and

$$\sum_{i=1}^{n-1} L_i L_{n-i} = nL_n - (F_n + 4F_{n-1}).$$

As we now demonstrate, both have rather nice generalizations for higher values of k. The convolution $\sum_{i=0}^{n} F_i F_{n-i} = (nL_n - F_n)/5$ also generalizes but in a not-so-nice way, and so we will not include it here.

This next theorem is rather dramatic since the right-hand side does not depend on k. In fact, it is identical to the convolution formula of the "regular" Fibonacci and Lucas numbers given above.

Theorem 5. *For $k \geq 2$ and $n \geq 1$, we have*

$$\sum_{i=1}^{n-1} F_i^{(k)} L_{n-i}^{(k)} = (n-1)F_n^{(k)}.$$

Proof. We know that the formula holds for $n = 1$ since the empty sum on the left would equal zero, and the $n-1$ on the right would equal zero as well. For $2 \leq n \leq k$, the left-hand side of our formula becomes $1 \cdot (2^{n-1}-1) + \sum_{i=2}^{n-1}(2^{i-2})(2^{n-i}-1)$, and a bit of arithmetic shows that this equals $(n-1)2^{n-2}$ which equals the right-hand side of our formula.

Next, for $n > k$ we suppose our formula holds for $n-1, n-2, \ldots, n-k$ and we use that to prove it is true at n. In other words, we assume that

$$\sum_{i=1}^{n-1-j} F_i^{(k)} L_{n-j-i}^{(k)} = (n-j-1)F_{n-j}^{(k)} \qquad \text{for } 1 \leq j \leq k. \tag{31}$$

Since $F_i^{(k)} = 0$ for $i \le 0$, we rewrite the left as $\sum_{i=1-j}^{n-1-j} F_i^{(k)} L_{n-j-i}^{(k)}$, which we re-index to obtain $\sum_{i=1}^{n-1} F_{i-j}^{(k)} L_{n-i}^{(k)}$. We now sum Equation (31) for j from 1 to k, and switch the order of summation on the left, to obtain

$$\sum_{i=1}^{n-1} \sum_{j=1}^{k} F_{i-j}^{(k)} L_{n-i}^{(k)} = \sum_{j=1}^{k} (n-j-1) F_{n-j}^{(k)}. \tag{32}$$

For the innner sum on the left of Equation (32), each $\sum_{j=1}^{k} F_{i-j}^{(k)}$ becomes $F_i^{(k)}$ for $i > 1$ and 0 for $i = 1$, so we actually can write the left as $\sum_{i=2}^{n-1} F_i^{(k)} L_{n-i}^{(k)}$. Meanwhile, on the right, we break this up into $\sum_{j=1}^{k} (n-1) F_{n-j}^{(k)}$ which equals $(n-1)F_n^{(k)}$, and $\sum_{j=1}^{k} -j F_{n-j}^{(k)}$ which equals $-L_{n-1}^{(k)}$ by Equation (5). We move the $L_{n-1}^{(k)}$ over to the left to obtain the desired formula. □

This next theorem on the convolution of the k-Lucas numbers has a surprisingly difficult proof despite its quite simple statement.

Theorem 6. *For $k \ge 2$ and $n \ge 1$, we have*

$$\sum_{i=1}^{n-1} L_i^{(k)} L_{n-i}^{(k)} = n\, L_n^{(k)} - \sum_{j=1}^{k} j^2 F_{n+1-j}^{(k)}. \tag{33}$$

Proof. We know that the formula holds for $n = 1$ since we would have zero on the left (from the empty sum) and zero on the right (since $L_1^{(k)} = F_1^{(k)} = 1$, and $F_i^{(k)} = 0$ for $i \le 0$). As for $2 \le n \le k$, the left-hand side of our formula becomes $\sum_{i=1}^{n-1} (2^i - 1)(2^{n-i} - 1)$, while the right-hand side becomes $n(2^n - 1) - \sum_{j=1}^{n-1} j^2\, 2^{n-1-j} - n^2$. A bit of arithmetic shows that these two sides are equal.

Next, for $n > k$ we suppose that our formula holds for $n-1, n-2, \dots, n-k$ and we use that to prove it is true at n. In other words, we assume that

$$\sum_{i=1}^{n-m-1} L_i^{(k)} L_{n-m-i}^{(k)} \;=\; (n-m) L_{n-m}^{(k)} - \sum_{j=1}^{k} j^2 F_{n-m+1-j}^{(k)} \tag{34}$$

holds true for $1 \le m \le k$. We now sum both sides of Equation (34) for m from 1 to k, giving us three expressions (one on the left of Equation (34) and the other two on the right). On the left we have

$$\sum_{m=1}^{k} \sum_{i=1}^{n-m-1} L_i^{(k)} L_{n-m-i}^{(k)}, \tag{35}$$

and on the right we have

$$\sum_{m=1}^{k} (n-m) L_{n-m}^{(k)}, \tag{36}$$

and also

$$-\sum_{m=1}^{k}\sum_{j=1}^{k} j^2 F^{(k)}_{n-m+1-j}. \tag{37}$$

This last expression is easy to deal with, since we can simply switch the summations and then use the recurrence formula to obtain

$$-\sum_{j=1}^{k} j^2 F^{(k)}_{n+1-j}, \tag{38}$$

which matches nicely with the last term in Equation (33). As for expression (36), we split this into two sums and then use the recurrence formula on the first to obtain

$$nL^{(k)}_n - \sum_{m=1}^{k} m\, L^{(k)}_{n-m}. \tag{39}$$

Finally, we turn our attention to expression (35), which will take some effort. We carefully split up the summation to obtain

$$\sum_{m=1}^{k}\sum_{i=1}^{n-k-1} L^{(k)}_i L^{(k)}_{n-m-i} \;+\; \sum_{m=1}^{k}\sum_{i=n-k}^{n-m-1} L^{(k)}_i L^{(k)}_{n-m-i}. \tag{40}$$

This first expression in (40) can be simplified to

$$\sum_{m=1}^{k}\sum_{i=1}^{n-k-1} L^{(k)}_i L^{(k)}_{n-m-i} \;=\; \sum_{i=1}^{n-k-1} L^{(k)}_i \sum_{m=1}^{k} L^{(k)}_{n-m-i} \;=\; \sum_{i=1}^{n-k-1} L^{(k)}_i L^{(k)}_{n-i}, \tag{41}$$

and for the second expression in (40) it is best to simply write it out as follows:

$$\begin{array}{rl}
(m=1) & L^{(k)}_{n-k}L^{(k)}_{k-1} + L^{(k)}_{n-k+1}L^{(k)}_{k-2} + \cdots + L^{(k)}_{n-4}L^{(k)}_{3} + L^{(k)}_{n-3}L^{(k)}_{2} + L^{(k)}_{n-2}L^{(k)}_{1} \\
(m=2) & L^{(k)}_{n-k}L^{(k)}_{k-2} + L^{(k)}_{n-k+1}L^{(k)}_{k-3} + \cdots + L^{(k)}_{n-4}L^{(k)}_{2} + L^{(k)}_{n-3}L^{(k)}_{1} \\
(m=3) & L^{(k)}_{n-k}L^{(k)}_{k-3} + L^{(k)}_{n-k+1}L^{(k)}_{k-4} + \cdots + L^{(k)}_{n-3}L^{(k)}_{1} \\
 & \vdots \qquad\qquad\quad \vdots \\
(m=k-1) & L^{(k)}_{n-k}L^{(k)}_{2} \quad + L^{(k)}_{n-k+1}L^{(k)}_{1} \\
(m=k) & L^{(k)}_{n-k}L^{(k)}_{1}.
\end{array}$$

Recall that $L^{(k)}_j = 2^j - 1$ for $1 \le j \le k$. This means that the entire first column in the above expression sums to $L^{(k)}_{n-k}(2^1 - 1 + 2^2 - 1 + 2^3 - 1 + \cdots + 2^{k-1} - 1)$ which simplifies to $L^{(k)}_{n-k}(2^k - 1 - k)$ which equals $L^{(k)}_{n-k}(L^{(k)}_k - k)$. Likewise, the second column simplifies to $L^{(k)}_{n-k+1}(L^{(k)}_{k-1} - (k-1))$, and on to the last column which we can write as $L^{(k)}_{n-2}(L^{(k)}_2 - 2)$. Writing this out, we have

$$\begin{aligned} L^{(k)}_{n-k}(L^{(k)}_k - k) + L^{(k)}_{n-k+1}(L^{(k)}_{k-1} - (k-1)) + \cdots + L^{(k)}_{n-2}(L^{(k)}_2 - 2) \\ = \sum_{i=n-k}^{n-2} L^{(k)}_i L^{(k)}_{n-i} - \sum_{m=2}^{k} m\, L^{(k)}_{n-m} \\ = \sum_{i=n-k}^{n-1} L^{(k)}_i L^{(k)}_{n-i} - \sum_{m=1}^{k} m\, L^{(k)}_{n-m}. \end{aligned} \tag{42}$$

We now have everything we need to finish the proof. Recall that expression (35) is equal to the sum of expressions (36) and (37). Replacing (35) with (41) and (42), and (36) with (39), and (37) with (38), we have

$$\begin{aligned} \sum_{i=1}^{n-k-1} L^{(k)}_i L^{(k)}_{n-i} + \sum_{i=n-k}^{n-1} L^{(k)}_i L^{(k)}_{n-i} - \sum_{m=1}^{k} m\, L^{(k)}_{n-m} \\ = nL^{(k)}_n - \sum_{m=1}^{k} m\, L^{(k)}_{n-m} - \sum_{j=1}^{k} j^2 F^{(k)}_{n+1-j}. \end{aligned}$$

This simplifies to our desired statement. □

5. Sums with Powers of 2

This first theorem generalizes the familiar formulas

$$\sum_{i=0}^{n} 2^{n-i} F_i = 2^{n+1} - F_{n+3}$$

and

$$\sum_{i=0}^{n} 2^{n-i} L_i = 3 \cdot 2^{n+1} - L_{n+3},$$

which come from Vajda [14].

Theorem 7. *For $k \geq 2$ and $n \geq 0$, we have*

$$\sum_{i=0}^{n} 2^{n-i} F^{(k)}_i \;=\; 2^{n+1} F^{(k)}_k - F^{(k)}_{n+k+1} \;=\; 2^{n+k-1} - F^{(k)}_{n+k+1},$$

and likewise

$$\sum_{i=0}^{n} 2^{n-i} L_i^{(k)} \;=\; 2^{n+1} L_k^{(k)} - L_{n+k+1}^{(k)} \;=\; 2^{n+1}\left(2^k - 1\right) - L_{n+k+1}^{(k)}.$$

Note that for $k = 3$ we can recover Adegoke's Tribonacci formula in [1].

Proof. We proceed by induction. The first formula at $n = 0$ is true since $F_{k+1}^{(k)} = 2^{k-1}$. For the second formula at $n = 0$, we know that $L_{k+1}^{(k)} = 2L_k^{(k)} - L_0^{(k)} = 2(2^k - 1) - k$, which means the second formula holds at $n = 0$.

Next, suppose each formula holds at n. We simply multiply each side by 2, add $F_{n+1}^{(k)}$ or $L_{n+1}^{(k)}$ to each side, and then use the equation $F_{n+k+2}^{(k)} = 2F_{n+k+1}^{(k)} - F_{n+1}^{(k)}$ to replace the $-2F_{n+k+1}^{(k)} + F_{n+1}^{(k)}$ with $-F_{n+k+2}^{(k)}$ (and likewise for the k-Lucas numbers) to obtain our desired formula. □

Our last theorem generalizes the delightful equation $\sum_{i=0}^{n} 2^i L_i = 2^{n+1} F_{n+1}$, which was first proved by Benjamin and Quinn [2] using colored tilings, and later by Sury [13] with polynomials in just a few lines, and then by Kwong [8] with generating functions in just a few words. What our generalization lacks in elegance, it makes up for with universality. We also give a strikingly similar version for the k-bonacci numbers.

Theorem 8. *For $k \geq 2$ and $n \geq 0$, we have*

$$\sum_{i=0}^{n} 2^i L_i^{(k)} \;=\; \frac{2^{n+1}}{2^{k+1} - 3}\left[3\sum_{j=1}^{k}\left(2^j - 1\right) L_{n+1+j}^{(k)} \;-\; (2^{k+2} - 6) L_{n+1+k}^{(k)} \;+\; \frac{2^{k+1} - 3k - 2}{2^{n+1}}\right],$$

and likewise

$$\sum_{i=0}^{n} 2^i F_i^{(k)} \;=\; \frac{2^{n+1}}{2^{k+1} - 3}\left[3\sum_{j=1}^{k}\left(2^j - 1\right) F_{n+1+j}^{(k)} \;-\; (2^{k+2} - 6) F_{n+1+k}^{(k)} \;-\; \frac{1}{2^n}\right].$$

Before venturing to the proof, we need this brief lemma.

Lemma 1. *For $S_n = \sum_{j=1}^{k} (2^j - 1) F_{n+1+j}^{(k)}$, then $2S_{n+1} - S_n = (2^{k+1} - 3) F_{n+k+2}^{(k)}$ for all $n \geq 1$.*

Proof of Lemma 1. We start with

$$2S_{n+1} - S_n = \sum_{j=1}^{k} (2^{j+1} - 2) F_{n+2+j}^{(k)} - \sum_{j=1}^{k} (2^j - 1) F_{n+1+j}^{(k)},$$

and if we replace j with $j-1$ in the first sum we obtain

$$2S_{n+1} - S_n = \sum_{j=2}^{k+1} (2^j - 2) F_{n+1+j}^{(k)} - \sum_{j=1}^{k} (2^j - 1) F_{n+1+j}^{(k)}.$$

When we combine these sums and adjust for the missing $j = 1$ term and the extra $j = k+1$ term in the first sum, we obtain

$$2S_{n+1} - S_n = \sum_{j=1}^{k} -F_{n+1+j}^{(k)} \; + 2^{k+1} F_{n+k+2}^{(k)} - 2F_{n+k+2}^{(k)}.$$

This first sum equals $-F_{n+k+2}^{(k)}$, so we obtain the desired $(2^{k+1} - 3)F_{n+k+2}^{(k)}$. □

Proof of Theorem 8. We will give a proof for the k-bonacci formula, and leave the k-Lucas formula as an exercise for the reader.

For $n = 0$, the left-hand side of our k-bonacci formula is zero, and inside the brackets on the right-hand side we have

$$\left[3 \sum_{j=1}^{k} \left(2^j - 1\right) 2^{j-1} \;-\; (2^{k+2} - 6) 2^{k-1} \;-\; 1 \right],$$

and a bit of arithmetic shows that this is zero as well.

For the induction step, we assume our formula is true at n and use it to prove that our formula is true at $n+1$. We rewrite our k-bonacci formula as

$$\sum_{i=0}^{n} 2^i F_i^{(k)} \;+\; \frac{2}{2^{k+1} - 3} \;=\; \frac{3 \cdot 2^{n+1}}{2^{k+1} - 3} S_n - 2^{n+2} F_{n+1+k}^{(k)}, \tag{43}$$

where $S_n = \sum_{j=1}^{k} (2^j - 1) F_{n+1+j}^{(k)}$ from Lemma 1.

And speaking of Lemma 1, we can use it to replace that S_n with $2S_{n+1} - (2^{k+1} - 3)F_{n+k+2}^{(k)}$, and so Equation (43) becomes

$$\sum_{i=0}^{n} 2^i F_i^{(k)} \;+\; \frac{2}{2^{k+1} - 3} = \frac{3 \cdot 2^{(n+1)+1}}{2^{k+1} - 3} S_{n+1} - 3 \cdot 2^{n+1} F_{n+k+2}^{(k)} - 2^{n+2} F_{n+1+k}^{(k)}.$$

We now add $2^{n+1} F_{n+1}^{(k)}$ to both sides. On the left, we now have a sum indexed from 0 to $n+1$. On the right, we use that $2^{n+1}(F_{n+1}^{(k)}) = 2^{n+1}(2F_{n+k+1}^{(k)} - F_{n+k+2}^{(k)})$. After a bit of simplification, our formula now reads

$$\sum_{i=0}^{n+1} 2^i F_i^{(k)} \;+\; \frac{2}{2^{k+1} - 3} = \frac{3 \cdot 2^{(n+1)+1}}{2^{k+1} - 3} S_{n+1} - 2^{(n+1)+2} F_{(n+1)+1+k}^{(k)},$$

which is our desired equation. □

6. Conclusion

As we have seen, summation and convolution formulas for the Fibonacci and Lucas numbers have surprisingly nice generalizations for the k-bonacci and k-Lucas numbers. In at least one case (Theorem 5) the formula is independent of k, which is rather unexpected. We can only imagine that there are many more delightful formulas just waiting to be discovered.

Acknowledgements. The authors would like to thank Pioneer Academics for making possible this research collaboration. The authors also express their gratitude to the anonymous referee for many helpful comments that greatly improved the quality of this paper.

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