A General Convolution Identity

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We will define all these numbers in a moment, but first let us write down some convolution formulas and see if any interesting patterns appear. We start with this well-known convolution formula [7] for the Fibonacci and Lucas numbers,

$$\sum_{i=0}^{n} F_i L_{n-i} \;=\; (n+1)F_n. \tag{1}$$

The Pell and Pell-Lucas numbers P_n, Q_n from Koshy's book [6] satisfy the somewhat similar equation

$$\sum_{i=1}^{n-1} P_i Q_{n-i} \;=\; \frac{1}{2}(n-1)P_n. \tag{2}$$

As for the Padovan and Perrin numbers A_n and E_n [9, 11, 13], we have

$$\sum_{i=0}^{n} A_i E_{n-i} \;=\; (n+5)A_n - E_{n+2}. \tag{3}$$

Finally, we have this unusual identity from Komatsu [5, Theorem 1] for the Tribonacci numbers T_n,

$$\sum_{i=0}^{n-3} T_i \left(T_{n-i} + T_{n-2-i} + 2T_{n-3-i}\right) \;=\; (n-2)T_{n-1} - T_{n-2}. \tag{4}$$

Although it might seem difficult to find a pattern in equations (1), (2), (3), and (4), there is indeed one single general convolution formula that holds for all the Fibonacci, Pell, Padovan, and Tribonacci numbers (and more), so long as we: choose the right initial values for our first sequence, define an appropriate companion sequence, and adjust the limits on the summation. The general formula will be

$$\sum_{i=0}^{n-1} \mathcal{F}_i \mathcal{L}_{n-i} \; = \; (n-1)\mathcal{F}_n, \tag{5}$$

where the numbers $\mathcal{F}_n$ represent any Fibonacci-type sequence such as the Fibonacci, Pell, Padovan, Tribonacci, etc., and likewise $\mathcal{L}_n$ represents its companion Lucas-type sequence; in equations (1), (2), and (3) that would be the Lucas, Pell-Lucas, and Perrin sequences, and in (4) it would be the numbers $T_n + T_{n-2} + 2T_{n-3}$. This universal convolution formula will produce equations (1), (2), (3), and (4), along with innumerable others. But before we formally state and prove our theorem, let us establish some definitions and provide some background material to motivate our subsequent arguments.

1 Definitions and Background

1.1 Fibonacci and Lucas numbers.

We recall that the Fibonacci numbers F_n are commonly defined as $F_0 = 0$ and $F_1 = 1$, with recurrence relation

$$F_n = F_{n-1} + F_{n-2} \qquad \text{for } n \geq 2, \tag{6}$$

giving us the sequence $0, 1, 1, 2, 3, 5, 8, 13, \ldots$. The closely-related Lucas numbers L_n have the same recurrence relation

$$L_n = L_{n-1} + L_{n-2} \qquad \text{for } n \geq 2 \tag{7}$$

as the Fibonaccis, but with the different starting values $L_0 = 2$ and $L_1 = 1$. This gives us the sequence $2, 1, 3, 4, 7, 11, 18, 29, \ldots$. For a deep and comprehensive review of these two sequences, we direct the reader to Koshy's book [7].

For a different approach to the Lucas numbers, we consider the elegant Binet formula involving the two roots of $x^2 - x - 1$,

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n. \tag{8}$$

It is rather surprising that this expression actually produces the Lucas numbers. Koshy [7, Chapter 5] gives a fairly standard induction proof, and Benjamin and Quinn [1, Identity 241] provide a delightful proof via probabilistic tilings.

1.2 Generating Functions.

For a different approach to the Fibonacci numbers, we consider the series

$$\frac{x}{1-x-x^2} \;=\; x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + \cdots \;=\; \sum_{n=0}^{\infty} F_n x^n. \tag{9}$$

It is rather surprising that this series actually has the Fibonacci numbers as coefficients, so let us spend a moment to convince ourselves of this. If we multiply the sum on the far right of equation (9) by $(1-x-x^2)$ and then distribute, we have

$$(1-x-x^2)\sum_{n=0}^{\infty} F_n x^n \;=\; \sum_{n=0}^{\infty} F_n x^n - \sum_{n=0}^{\infty} F_n x^{n+1} - \sum_{n=0}^{\infty} F_n x^{n+2}, \tag{10}$$

and if we now re-index the second and third sum on the right we have

$$(1-x-x^2)\sum_{n=0}^{\infty} F_n x^n \;=\; \sum_{n=0}^{\infty} F_n x^n - \sum_{n=1}^{\infty} F_{n-1} x^n - \sum_{n=2}^{\infty} F_{n-2} x^n \tag{11}$$

$$= F_0 + F_1 x - F_0 x + \sum_{n=2}^{\infty} \left(F_n - F_{n-1} - F_{n-2}\right) x^n$$

$$= 0 + 1x - 0x + \sum_{n=2}^{\infty} (\,0\,)\, x^n \;=\; x, \tag{12}$$

where that last line follows from our initial values $F_0 = 0$ and $F_1 = 1$ and our recurrence relation (6) for $n \geq 2$. If we divide both sides by $(1-x-x^2)$ we obtain our formula (9), as desired.

Of course, what we are doing here is re-discovering the concept of *generating functions*, which are functions like $x/(1-x-x^2)$ whose series expansion has coefficients of interest. To be precise, we would say that $x/(1-x-x^2)$ is a *generating function for the Fibonacci numbers* since the expansion of $x/(1-x-x^2)$ in equation (9) *generates* the Fibonaccis as coefficients. Likewise, it is not hard to show that $(2-x)/(1-x-x^2)$ is a generating function for the Lucas numbers. We do not concern ourselves with issues of convergence for these power series; we simply use these generating functions as representations of formal power series that we can then multiply, integrate, or differentiate in order to prove various identities. For example, we can use them to prove equation (1) on the convolution of the Fibonacci and Lucas numbers, and it is worth our while to sketch out this technique as we will be using the same method in greater generality a bit later when we prove Theorem 1.

On the one hand, we can multiply our two power series as follows,

$$\left(\frac{x}{1-x-x^2}\right)\left(\frac{2-x}{1-x-x^2}\right) = \left(\sum_{n=0}^{\infty} F_n x^n\right)\left(\sum_{n=0}^{\infty} L_n x^n\right)$$

$$= \sum_{n=0}^{\infty} \Big(F_n L_0 + F_{n-1} L_1 + \cdots + F_0 L_n \Big) x^n,$$

and on the other hand, with some work we can show that

$$\begin{aligned} \left(\frac{x}{1-x-x^2} \right) \left(\frac{2-x}{1-x-x^2} \right) &= \frac{d}{dx} \left(\frac{x^2}{1-x-x^2} \right) \\ &= \frac{d}{dx} \left(\sum_{n=0}^{\infty} F_n x^{n+1} \right) = \sum_{n=0}^{\infty} (n+1) F_n x^n \end{aligned}$$

(we leave the details to the reader) and so by comparing the coefficients of the sums on the right of the above equations, we have

$$F_n L_0 + F_{n-1} L_1 + \cdots + F_0 L_n = (n+1) F_n.$$

This is the technique that we will use in what follows, and the truly surprising part is that this procedure with generating functions will apply not just to the Fibonacci and Lucas numbers but also to all the other sequences discussed at the beginning of this article. For a deep and comprehensive study of generating functions, we direct the reader to Wilf's book [14].

1.3 Pell, Padovan, Tribonacci, and more.

Moving on to the other sequences mentioned in the introduction, we note that Koshy [6] defined the Pell numbers P_n as having initial values $P_1 = 1$ and $P_2 = 2$, and then satisfying the recurrence relation

$$P_n = 2P_{n-1} + P_{n-2}. \tag{13}$$

This sequence, beginning at $n = 1$, is $1, 2, 5, 12, 29, 70, \ldots$. Koshy defined the Pell-Lucas numbers Q_n to satisfy the same recurrence as the Pell numbers but with different initial values $Q_1 = 1$ and $Q_2 = 3$. This sequence, again starting at $n = 1$, is $1, 3, 7, 17, 41, 99, \ldots$.

Stewart [11] defined the Padovan and Perrin numbers A_n and E_n by the recurrences

$$A_n = A_{n-2} + A_{n-3} \qquad \text{and} \qquad E_n = E_{n-2} + E_{n-3}, \tag{14}$$

with initial values $A_0 = A_1 = A_2 = 1$ and $E_0 = 3, E_1 = 0, E_2 = 2$. These sequences, starting at $n = 0$, are $1, 1, 1, 2, 2, 3, 4, \ldots$ and $3, 0, 2, 3, 2, 5, 5, \ldots$ respectively.

For the Tribonacci numbers, Komatsu [5] defines them in the standard way as satisfying the recurrence relation

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} \tag{15}$$

with initial values $T_0 = 0$, $T_1 = 1$, and $T_2 = 1$, and so starting at $n = 0$ we have $0, 1, 1, 2, 4, 7, 13, \ldots$. If we set

$$U_n = T_n + T_{n-2} + 2T_{n-3} \quad \text{for } n \geq 3, \tag{16}$$

then Komatsu's unusual convolution formula (4) becomes

$$\sum_{i=0}^{n-3} T_i\, U_{n-i} \;=\; (n-2)T_{n-1} - T_{n-2}. \tag{17}$$

From our definition of U_n in (16), we see that it has "initial" values $U_3 = 3, U_4 = 7$, and $U_5 = 11$, and that U_n satisfies the same recurrence relation as the Tribonaccis in (15),

$$U_n = U_{n-1} + U_{n-2} + U_{n-3}. \tag{18}$$

From this we can work backwards and define $U_2 = 1$, and then $U_1 = 3$. This gives us the sequence $3, 1, 3, 7, 11, 21, \ldots$ for U_n starting at $n = 1$.

1.4 Newton's Identities.

Speaking of Tribonacci numbers, Yilmaz and Taskara [15] defined the Tribonacci-Lucas numbers W_n as

$$W_n \;=\; (\theta_1)^n + (\theta_2)^n + (\theta_3)^n, \tag{19}$$

where θ_1, θ_2, and θ_3 are the three solutions to $x^3 - x^2 - x - 1 = 0$. This reminds us of the Binet formula (8) for the "original" Lucas numbers L_n. It is rather surprising that the definition of W_n in (19) produces integer values; it is perhaps even more surprising to learn that $W_n = U_{n+1}$ with U_n from equations (16) and (17). It is worth our while to verify this, as we will be using this same idea in greater generality a bit later when we define the Lucas-type numbers $\mathcal{L}_n$.

Since θ_1, θ_2, and θ_3 are the three roots of $x^3 - x^2 - x - 1$, then this means that

$$\begin{aligned} x^3 - x^2 - x - 1 &= (x-\theta_1)(x-\theta_2)(x-\theta_3) && (20)\\ &= x^3 - (\theta_1+\theta_2+\theta_3)x^2 \\ &\quad + (\theta_1\theta_2 + \theta_2\theta_3 + \theta_3\theta_1)x - (\theta_1\theta_2\theta_3), && (21) \end{aligned}$$

and so without ever calculating the values of θ_1, θ_2, or θ_3 we can simply compare the coefficients in the above equations to see that

$$\begin{aligned} \theta_1 + \theta_2 + \theta_3 &= 1 \\ \theta_1\theta_2 + \theta_2\theta_3 + \theta_3\theta_1 &= -1 \\ \theta_1\theta_2\theta_3 &= 1 \end{aligned}$$

and hence $W_1 = 1$. For the value of W_2, we note that

$$\begin{aligned} (\theta_1)^2 + (\theta_2)^2 + (\theta_3)^2 \;&=\; (\theta + \theta_2 + \theta_3)^2 - 2(\theta_1\theta_2 + \theta_2\theta_3 + \theta_3\theta_1) && (22)\\ &=\; (1)^2 - 2(-1) \;=\; 3 && (23) \end{aligned}$$

and so $W_2 = 3$.

If we now wanted to calculate W_3, we could note that since each θ_i is a root of $x^3 - x^2 - x - 1$ then each θ_i satisfies

$$(\theta_i)^3 = (\theta_i)^2 + (\theta_i)^1 + 1,$$

and so if we sum the above equation from $i = 1$ to 3 we obtain

$$W_3 = W_2 + W_1 + 3,$$

telling us that $W_3 = 3 + 1 + 3 = 7$. Of course, what we are doing here is re-discovering *Newton's identities*, which state that "power sums" like $W_n = (\theta_1)^n + (\theta_2)^n + (\theta_3)^n$ can always be written in terms of the coefficients of the polynomial x^3-x^2-x-1 and the power sums of lesser degree, as seen above. To be precise, here is what Kalman's version [4] of Newton's identities says about W_n:

$$\begin{aligned}
&W_1 = 1, \\
&W_2 = 1 \cdot W_1 + 2 \ = \ 3, \\
&W_3 = 1 \cdot W_2 + 1 \cdot W_1 + 3 \ = \ 7, \\
\text{and} \quad &W_n = 1 \cdot W_{n-1} + 1 \cdot W_{n-2} + 1 \cdot W_{n-3} \qquad \text{for } n > 3.
\end{aligned}$$

This last equation is a perfect match for the recurrence relation for U_n in equation (18), and by checking the initial values for U_n and W_n we can easily convince ourselves that $W_n = U_{n+1}$. We will give a more general version of Newton's identities in a moment.

2 Fibonacci-type and Lucas-type sequences

At this point, we are ready to define what we mean by a "Fibonacci-type" and "Lucas-type" sequence. Given a fixed k and a fixed list of integers $c_1, c_2, \ldots, c_k$ with $c_k \neq 0$, we define the associated *Fibonacci-type sequence* $\mathcal{F}_n$ as

$$\begin{aligned}
\mathcal{F}_n &= 0 \quad \text{for all } n < 0, \\
\mathcal{F}_0 &= 1, \\
\text{and} \quad \mathcal{F}_n &= c_1\mathcal{F}_{n-1} + c_2\mathcal{F}_{n-2} + \cdots + c_k\mathcal{F}_{n-k} \quad \text{for all } n \geq 2.
\end{aligned} \tag{24}$$

As for the companion *Lucas-type sequence* $\mathcal{L}_n$, we will define it in terms of the k roots $\theta_1, \theta_2, \ldots, \theta_k$ of the characteristic polynomial

$$x^k - \left(c_1x^{k-1} + c_2x^{k-2} + \cdots + c_k\right) \tag{25}$$

as follows:

$$\mathcal{L}_n \ = \ (\theta_1)^n + (\theta_2)^n + \cdots + (\theta_k)^n. \tag{26}$$

We note that by direct calculation we have $\mathcal{L}_0 = k$. As for the other values of $\mathcal{L}_n$, we call once more upon the elegant presentation of Newton's identities in Kalman's paper [4], which gives us the tidy little formulas

$$\mathcal{L}_n = c_1\mathcal{L}_{n-1} + c_2\mathcal{L}_{n-2} + \cdots + c_{n-1}\mathcal{L}_1 \ + \ n\,c_n \qquad \text{for } 1 \leq n \leq k, \tag{27}$$

$$\mathcal{L}_n \;=\; c_1\mathcal{L}_{n-1} + c_2\mathcal{L}_{n-2} + \cdots + c_k\mathcal{L}_{n-k} \qquad\qquad \text{for } n > k. \tag{28}$$

Together, these equations reassure us that $\mathcal{L}_n$ is always an integer, and from equation (28) we see that $\mathcal{L}_n$ satisfies exactly the same recurrence as $\mathcal{F}_n$ from equation (24).

After we state and prove our theorem, we will then show that our definition of $\mathcal{F}_n$ in equation (24) will match nicely with the Fibonacci, Pell, Padovan, and Tribonacci numbers from equations (1), (2), (3), and (4) so long as we choose the appropriate constants c_1, c_2 or c_1, c_2, c_3 and perhaps adjust the indexing if necessary. Likewise, we will show that our definition of $\mathcal{L}_n$ in equation (26) will give us the Lucas, Pell-Lucas, Perrin, and Tribonacci-Lucas numbers from the same equations (again with some adjustment as needed). We have already seen this at work for the Lucas numbers L_n in equation (8) and for the Tribonacci-Lucas numbers W_n in equation (19).

Benjamin and Quinn [1, §3.1] provide an alternate approach to defining $\mathcal{F}_n$ and $\mathcal{L}_n$ which works well for $c_1, c_2, \ldots$ non-negative integers. They define $\mathcal{F}_n$ to be the number of ways to tile a strip of length $n-1$ with c_1 colors of squares, c_2 colors of dominos, and so on, and likewise $\mathcal{L}_n$ for tiling a bracelet of length n, with the additional stipulations that $\mathcal{F}_1 = 1$ and $\mathcal{L}_0 = k$. It is possible to show that this approach would give us the same sequences as from our earlier definitions. A third approach would be to define these sequences in terms of their generating functions, which we give in the proof of Theorem 1, below. We feel that our approach of using initial values to define $\mathcal{F}_n$ and a Binet-type formula to define $\mathcal{L}_n$ is the most natural of these three options.

3 Theorem and Proof

We are now ready to state and prove our main theorem.

Theorem 1. *For any Fibonacci-type sequence of numbers $\mathcal{F}_n$ and companion Lucas-type sequence of numbers $\mathcal{L}_n$ as defined above in equations (24) and (26), we have*

$$\sum_{i=0}^{n-1} \mathcal{F}_i\mathcal{L}_{n-i} \;=\; (n-1)\mathcal{F}_n. \tag{29}$$

Proof. Our proof will have three parts. First, we will show that the **generating functions** $f(x)$ and $\ell(x)$ for the numbers $\mathcal{F}_n$ and $\mathcal{L}_n$ are

$$f(x) = \sum_{n=0}^{\infty} \mathcal{F}_n x^n = \frac{x}{h(x)} \qquad \text{and} \qquad \ell(x) = \sum_{n=0}^{\infty} \mathcal{L}_n x^n = k - \frac{xh'(x)}{h(x)} \tag{30}$$

respectively, with $h(x)$ defined as

$$h(x) = 1 - \left(c_1 x + c_2 x^2 + \cdots + c_k x^k\right). \tag{31}$$

Then, we will show that the **product** $f(x)\ell(x)$ equals $(k-1)f(x) + xf'(x)$. And finally, we will use that equality to establish our **convolution equation** (29).

First, the **generating functions**. To show that $f(x)$ in equation (30) really is the same as $x/h(x)$, we will look at $f(x)h(x)$ and show that it equals x. To do this, we note that since we defined $\mathcal{F}_n = 0$ for $n < 0$ in (24), then we can extend our sum in (30) as follows:

$$f(x) = \sum_{n=0}^{\infty} \mathcal{F}_n x^n = \sum_{n=-\infty}^{\infty} \mathcal{F}_n x^n.$$

To calculate $f(x)h(x)$, we bring $h(x)$ inside the sum in the above equation to give us

$$f(x)h(x) = \sum_{n=-\infty}^{\infty} \mathcal{F}_n x^n (1 - c_1 x - c_2 x^2 - \cdots - c_k x^k).$$

By splitting apart this sum and then re-indexing each individual sum, just as we did in equations (10) and (11), this all becomes

$$f(x)h(x) = \sum_{n=-\infty}^{\infty} \Big(\mathcal{F}_n - c_1 \mathcal{F}_{n-1} - c_2 \mathcal{F}_{n-2} - \cdots - c_k \mathcal{F}_{n-k}\Big) x^n.$$

Now, for $n \geq 2$ the coefficient of x^n in the above sum will vanish thanks to the last part of equation (24), and likewise for $n < 0$ by the first part of equation (24). Finally, for $n = 1$ the coefficient of x^1 in the above sum has just one non-zero term and that is $\mathcal{F}_1 = 1$. We conclude that

$$f(x)h(x) \;=\; \Big(\mathcal{F}_1\Big) x^1 = x$$

and so $f(x) = x/h(x)$ as desired.

We pause here to explain how we knew that $x/h(x)$ would be the generating function for the sequence of Fibonacci-type numbers $\mathcal{F}_n$ and likewise $k - xh'(x)/h(x)$ for $\mathcal{L}_n$: it was all due to trial and error. We knew that $x/(1 - x - x^2)$ was the generating function for the Fibonaccis, and we guessed correctly that $x/(1-x-x^2-x^3)$ would give the Tribonaccis. From this point, it took not much effort to arrive at $x/h(x)$ as the general form. It took quite a bit of effort to arrive at $k-xh'(x)/h(x)$ for $\mathcal{L}_n$; we started with $(2-x)/(1-x-x^2)$ which generates the Lucas numbers and then did the same for innumerable other sequences until the pattern popped into view. We also note that when Komatsu [5] proved equation (4) for the Tribonaccis, he used the generating function $x/(1-x-x^2-x^3)$ as well as its derivative, and so that gave us a hint that we should be thinking about derivatives in the general case.

Moving on, we still need to demonstrate that $\ell(x)$ in equation (30) really is the same as $k - xh'(x)/h(x)$. We will look at $\ell(x)h(x)$ and show that it equals $kh(x) - xh'(x)$, and to do this, we write out $\ell(x)h(x)$, bringing $h(x)$ inside the sum, to give us

$$\ell(x)h(x) = \sum_{n=0}^{\infty} \mathcal{L}_n x^n (1 - c_1 x - c_2 x^2 - \cdots - c_k x^k).$$

If we write out all the terms on the right, gathering together those terms with the same power of x, this all becomes

$$\begin{aligned}\ell(x)h(x) &= \mathcal{L}_0 + \Big(\mathcal{L}_1 - c_1\mathcal{L}_0\Big)x + \Big(\mathcal{L}_2 - c_1\mathcal{L}_1 - c_2\mathcal{L}_0\Big)x^2 \\ &\quad + \Big(\mathcal{L}_3 - c_1\mathcal{L}_2 - c_2\mathcal{L}_1 - c_3\mathcal{L}_0\Big)x^3 + \cdots .\end{aligned}$$

For $1 \le n \le k$. the coefficient of x^n in the above sum is

$$\Big(\mathcal{L}_n - c_1\mathcal{L}_{n-1} - c_2\mathcal{L}_{n-2} - \cdots - c_{n-1}\mathcal{L}_1 - c_n\mathcal{L}_0\Big)$$

and thanks to the first Newton's identity (27) this equals $nc_n - c_n\mathcal{L}_0$. For $n > k$, the coefficient of x^n is

$$\Big(\mathcal{L}_n - c_1\mathcal{L}_{n-1} - c_2\mathcal{L}_{n-2} - \cdots - c_k\mathcal{L}_{n-k}\Big)$$

and thanks to the second Newton's identity (28) this equals 0. So, we have

$$\ell(x)h(x) = \mathcal{L}_0 + \sum_{n=1}^{k} \Big(nc_n - c_n\mathcal{L}_0\Big)x^n,$$

and if we split apart the sum and write everything out, we have

$$\begin{aligned}\ell(x)h(x) &= \mathcal{L}_0 + \Big(1c_1x + 2c_2x^2 + 3c_3x^3 + \cdots + kc_kx^k\Big) \\ &\quad - \mathcal{L}_0\Big(c_1x + c_2x^2 + c_3x^3 + \cdots + c_kx^k\Big)\end{aligned}$$

We can see that the second line in the above equation is $-\mathcal{L}_0\Big(1 - h(x)\Big)$ thanks to our definition of $h(x)$ in equation (31). It is a bit harder to see that the first line will be $\mathcal{L}_0 + \Big(-xh'(x)\Big)$, but if we calculate $-xh'(x)$ we can see the connection. Hence, we have shown

$$\ell(x)h(x) = \mathcal{L}_0 + \Big(-xh'(x)\Big) - \mathcal{L}_0\Big(1 - h(x)\Big),$$

and since $\mathcal{L}_0 = k$ then this last line simplifies to $kh(x) - xh'(x)$, as desired.

Next, the **product** $f(x)\ell(x)$. Since our equation for $\ell(x)$ in (30) has the expression $h'(x)/h(x)$, it is reasonable to start with something involving $\ln h(x)$ and then take the derivative to obtain that $h'(x)/h(x)$. After a considerable amount of trial and error (again), we arrived at the following approach: we start with $f(x) = x/h(x)$ from (30), and we take the natural logarithm to obtain $\ln f(x) = \ln x - \ln h(x)$. Next, taking the derivative of both sides, we find that

$$\frac{f'(x)}{f(x)} = \frac{1}{x} - \frac{h'(x)}{h(x)}.$$

If we multiply both sides by x, we obtain

$$\frac{xf'(x)}{f(x)} = 1 - \frac{xh'(x)}{h(x)},$$

and if we add $k-1$ to both sides and simplify, we have

$$\frac{(k-1)f(x) + xf'(x)}{f(x)} = k - \frac{xh'(x)}{h(x)}, \tag{32}$$

and we recognize that the right-hand side of equation (32) is $\ell(x)$, so after multiplying both sides by $f(x)$ we obtain our desired equation,

$$(k-1)f(x) + xf'(x) = f(x)\ell(x). \tag{33}$$

Finally, the **convolution equation**. We recall that $f(x) = \sum \mathcal{F}_n x^n$ and $\ell(x) = \sum \mathcal{L}_n x^n$, and so equation (33) becomes

$$(k-1)\sum_{n=0}^{\infty} \mathcal{F}_n x^n + x\frac{d}{dx}\sum_{n=0}^{\infty} \mathcal{F}_n x^n \;=\; \sum_{n=0}^{\infty} \mathcal{F}_n x^n \sum_{n=0}^{\infty} \mathcal{L}_n x^n. \tag{34}$$

After simplifying the left-hand side of the above equation, we have

$$\sum_{n=0}^{\infty} \Big((k-1)\mathcal{F}_n + n\mathcal{F}_n\Big)x^n \;=\; \sum_{n=0}^{\infty} \mathcal{F}_n x^n \sum_{n=0}^{\infty} \mathcal{L}_n x^n, \tag{35}$$

and if we now multiply out the two series on the right of the above equation, we will find that we have

$$\sum_{n=0}^{\infty} \Big((k-1)\mathcal{F}_n + n\mathcal{F}_n\Big)x^n \;=\; \sum_{n=0}^{\infty} \Big(\mathcal{F}_n\mathcal{L}_0 + \cdots + \mathcal{F}_0\mathcal{L}_n\Big)x^n. \tag{36}$$

Comparing the coefficients of x^n in the above equation, we have

$$(k-1)\mathcal{F}_n + n\mathcal{F}_n \;=\; \sum_{i=0}^{n} \mathcal{F}_i\mathcal{L}_{n-i}. \tag{37}$$

If we now subtract the last term in that summation from both sides, we have

$$(k-1)\mathcal{F}_n + n\mathcal{F}_n - \mathcal{F}_n\mathcal{L}_0 \;=\; \sum_{i=0}^{n-1} \mathcal{F}_i\mathcal{L}_{n-i}, \tag{38}$$

and since $\mathcal{L}_0 = k$ then the left-hand side of (38) simplifies to $(n-1)\mathcal{F}_n$, thus giving us our desired equation (29). □

4 Examples

We can now use equation (29) in Theorem 1 to establish equations (1), (2), (3), and (4) from the beginning of this paper.

First, we address equation (1) with the Fibonacci and Lucas numbers. If we set $k = 2$ and coefficients $c_1 = c_2 = 1$, then the Fibonacci-type numbers $\mathcal{F}_n$ from (24) are a perfect match for our "regular" Fibonaccis F_n from (6), and likewise the Lucas-type numbers $\mathcal{L}_n$ from (26) match up just fine with the "regular" Lucas numbers L_n in equation (8). Thus, equation (29) becomes

$$\sum_{i=0}^{n-1} F_i \, L_{n-i} \;=\; (n-1)F_n,$$

and if we add $F_n L_0 = 2F_n$ to both sides we obtain equation (1).

Likewise, for equation (2) with the Pell and Pell-Lucas numbers, we set $k = 2$ and we assign $c_1 = 2$ and $c_2 = 1$. The definition of $\mathcal{F}_n$ in this case from (24) would be $\mathcal{F}_0 = 0, \mathcal{F}_1 = 1$, and $\mathcal{F}_n = 2\mathcal{F}_{n-1} + \mathcal{F}_{n-2}$, and this matches Koshy's definition of the Pell numbers P_n in (13) so long as we keep $n \geq 1$. As for the Lucas-type numbers $\mathcal{L}_n$ in this case, the definition in (26) along with Newton's identities (27) and (28) give us that $\mathcal{L}_0 = 2$, $\mathcal{L}_1 = 2$, and $\mathcal{L}_n = 2\mathcal{L}_{n-1} + \mathcal{L}_{n-2}$. This is the sequence $2, 2, 6, 14, 34, 82 \ldots$ starting at $n = 0$, and when we compare it to Koshy's sequence Q_n from earlier which is $1, 3, 7, 17, 41, \ldots$ starting at $n = 1$ and with the same recurrence $Q_n = 2Q_{n-1} + Q_{n-2}$, we see that $\mathcal{L}_n = 2Q_n$ so long as $n \geq 1$.

So, keeping in mind that Koshy's P_n and Q_n are only defined for $n \geq 1$, we can re-write equation (29) in this case to give us

$$\mathcal{F}_0\mathcal{L}_n + \sum_{i=1}^{n-1} P_i \, 2Q_{n-i} \;=\; (n-1)P_n,$$

and if we recall that $\mathcal{F}_0 = 0$ and if we divide both sides by 2 we obtain equation (2).

As for equation (3) with the Padovan and Perrin numbers, their definitions in equation (14) tells us that we should use $k = 3$ and we should set $c_1 = 0$ and $c_2 = c_3 = 1$. When we write out the numbers $\mathcal{F}_n$ in this case, we have (starting at $n = 0$) the sequence $0, 1, 0, 1, 1, 1, 2, 2, 3, \ldots$, and when we compare this to Stewart's definition of the Padovan numbers A_n from earlier we see that $\mathcal{F}_n = A_{n-3}$ (for $n \geq 3$). As for the numbers $\mathcal{L}_n$ in this case, from equations (26), (27), and (28) we have, starting at $n = 0$, the sequence $3, 0, 2, 3, 2, 5, 5, \ldots$ which is a perfect match for the Perrin numbers, giving us $\mathcal{L}_n = E_n$.

So, to derive equation (3) we begin with equation (29) but with $n+3$ instead of n and we separate out the first three terms from that sum to obtain

$$\mathcal{F}_0\mathcal{L}_{n+3} + \mathcal{F}_1\mathcal{L}_{n+2} + \mathcal{F}_2\mathcal{L}_{n+1} + \sum_{i=3}^{n+2} \mathcal{F}_i\mathcal{L}_{(n+3)-i} \;=\; (n+2)\mathcal{F}_{n+3}.$$

If we now substitute $\mathcal{F}_0 = 0$, $\mathcal{F}_1 = 1$, $\mathcal{F}_2 = 0$, $\mathcal{F}_n = A_{n-3}$ (this last one for $n \geq 3$) and $\mathcal{L}_n = E_n$, then the above equation becomes

$$E_{n+2} + \sum_{i=3}^{n+2} A_{i-3}E_{n+3-i} \;=\; (n+2)A_n,$$

and if we re-index the summation then this becomes

$$E_{n+2} + \sum_{i=0}^{n-1} A_i E_{n-i} \;=\; (n+2)A_n.$$

It remains only to move the E_{n+2} to the other side, and add $A_n E_0 = 3A_n$ to both sides, to obtain equation (3).

Finally, we have the Tribonacci convolution in equation (4). Thanks to the recursion for the Tribonacci numbers in equation (15) we know we should set $k = 3$ and $c_1 = c_2 = c_3 = 1$. The Fibonacci-type numbers $\mathcal{F}_n$ in this case, from (24), match perfectly with T_n. As for $\mathcal{L}_n$ in this case, the definition of $\mathcal{L}_n$ in (26) is identical to the definition of W_n in (19), and we already discovered that these numbers W_n satisfy $W_n = U_{n+1}$, so we have that $\mathcal{L}_n = U_{n+1}$. Thus, to establish (4) we should begin with equation (29) but with $n-1$ instead of n, and we then separate out the last term in the sum to obtain

$$\sum_{i=0}^{n-2} \mathcal{F}_i\mathcal{L}_{(n-1)-i} \;=\; \sum_{i=0}^{n-3} \mathcal{F}_i\mathcal{L}_{(n-1)-i} + \mathcal{F}_{n-2}\mathcal{L}_1 \;=\; (n-2)\mathcal{F}_{n-1}.$$

We can now use the identities $\mathcal{L}_1 = 1$, $\mathcal{F}_n = T_n$, and $\mathcal{L}_n = U_{n+1}$ to quickly obtain

$$\sum_{i=0}^{n-3} T_i U_{n-i} + T_{n-2} \;=\; (n-2)T_{n-1},$$

and this gives us equation (17) which, thanks to our definition of U_n, is also equation (4).

5 Conclusion

We point out that many other authors have studied the convolution of Fibonacci-type and Lucas-type numbers. Zeitlin [16, Formula (5.6)] gave a version of Theorem 1 for second-order recurrences; his formula is

$$\sum_{i=0}^{n} Z_{n+1-i}V_{i+1} \;=\; (n+1)Z_{n+2}, \tag{39}$$

where his Z_n is the Fibonacci-type sequence that satisfies the recurrence formula $Z_n = aZ_{n-1}+bZ_{n-2}$, and likewise his V_n is the Lucas-type sequence that satisfies the same recurrence. Robbins [10, Theorem 5] improved on Zeitlin's result with an identity that holds for weighted convolutions of Zeitlin's Z_n and V_n sequences. Szakács [12, Theorem 5] also covered second-order recurrences and gave specific examples for the Fibonacci, Pell, and Jacobsthal sequences. Dresden and Wang [2, Theorem 5] gave a induction proof of a version of Theorem 1 that applies to the so-called k-bonacci numbers. Ours is the only result that applies to *any* recurrence sequence $\mathcal{F}_n$ so long as $\mathcal{F}_1 = 1$ and $\mathcal{F}_n = 0$ for $n \leq 0$.

Finally, we note that there is plenty of work still to be done in convolutions. First, here is a formula that can be derived from our Theorem 1 using $k = 3$ with $c_1 = c_2 = 2$ and $c_3 = -1$:

$$\sum_{i=0}^{n} F_i F_{i+1} F_{n-i}^2 \;=\; \frac{n+2}{5} F_n F_{n+1} - \frac{3}{25}\Big(F_{2n+2} + (-1)^{n+1}(n+1)\Big).$$

We leave the (quite challenging!) details to the reader, along with a challenge to find other convolution formulas like this.

And second, here are two additional convolution formulas that have no connection to our Theorem 1 yet still show a remarkable pattern. Koshy and Griffiths [8, equation (2.2)] discovered this delightful convolution formula that connects the seemingly-unrelated Jacobsthal numbers J_n and Fibonacci numbers F_n,

$$\sum_{i=0}^{n} J_i F_{n-i} = J_{n+1} - F_{n+1}, \tag{40}$$

and both Frontczak [3, Theorem 2.1] and Benjamin and Quinn [1, p. 47] have found an equally delightful formula that links the Tribonacci numbers T_n with the Fibonacci numbers F_n,

$$\sum_{i=0}^{n} T_i F_{n-i} = T_{n+2} - F_{n+2}. \tag{41}$$

We can't help but notice the similarity between equations (40) and (41), and we can't help but ask if there are other general convolution formulas waiting to be discovered.

References

[1] Benjamin, A., Quinn, J. (2003). *Proofs that Really Count.* Washington, DC: MAA Press.

[2] Dresden, G., Wang, Y. (2021). Sums and convolutions of k-bonacci and k-Lucas numbers. *Integers.* 21: Paper No. A56.

[3] Frontczak, R. (2018). Some Fibonacci-Lucas-Tribonacci-Lucas identities. *Fibonacci Quart.* 56(3): 263–274.

[4] Kalman, D. (2000). A matrix proof of Newton's identities. *Math. Mag.* 73(4): 313–315.

[5] Komatsu, T. (2018). Convolution identities for tribonacci numbers. *Ars Combin.* 136: 199–210.

[6] Koshy, T. (2014). *Pell and Pell-Lucas numbers with applications.* New York: Springer. doi.org/10.1007/978-1-4614-8489-9

[7] Koshy, T. (2018). *Fibonacci and Lucas numbers with applications. Vol. 1.* Hoboken, NJ: John Wiley & Sons, Inc.

[8] Koshy, T., Griffiths, M. (2018). Some gibonacci convolutions with dividends. *Fibonacci Quart.* 56(3): 237–245.

[9] Rihane, S. E., Adegbindin, C. A., Togbé, A. (2020). Fermat Padovan and Perrin numbers. *J. Integer Seq.* 23(6): Article 20.6.2.

[10] Robbins, N. (1991). Some convolution-type and combinatorial identities pertaining to binary linear recurrences. *Fibonacci Quart.* 29(3): 249–255.

[11] Stewart, I. (1996). Tales of a neglected number. *Sci. Amer.* 274(6): 102–103.

[12] Szakács, T. (2017). Convolution of second order linear recursive sequences II. *Commun. Math.* 25(2): 137–148. doi.org/10.1515/cm-2017-0011

[13] Tedford, S. J. (2019). Combinatorial identities for the Padovan numbers. *Fibonacci Quart.* 57(4): 291–298.

[14] Wilf, H. S. (2006). *generatingfunctionology.* Wellesley, MA: A K Peters, Ltd.

[15] Yilmaz, N., Taskara, N. (2014). Tribonacci and Tribonacci-Lucas numbers via the determinants of special matrices. *Appl. Math. Sci. (Ruse).* 8(37-40): 1947–1955. doi.org/10.12988/ams.2014.4270

[16] Zeitlin, D. (1967). On convoluted numbers and sums. *Amer. Math. Monthly.* 74: 235–246. doi.org/10.2307/2316014