Proposed Problem.

Gregory Dresden (Washington & Lee University, Lexington, VA, USA), and
Yunqi Liu (Nanjing Foreign Language School, Nanjing, China).

PROBLEM:

Suppose $\phi > 0$ satisfies the equation $\phi^a + \phi^b = 2$ for distinct integers a, b. Show that every natural number n can be written as a sum of exactly n distinct integer powers of ϕ.

For example, the golden ratio $\phi = (1+\sqrt{5})/2$ satisfies $\phi^{-2} + \phi^1 = 2$, and the natural number $n = 4$ can indeed be written as a sum of four distinct powers of ϕ, as shown here:

$$4 = \phi^{-4} + \phi^{-3} + \phi^0 + \phi^2.$$

Likewise, the tribonacci ratio γ satisfies $\gamma^{-3} + \gamma^1 = 2$, and we can write 7 as a sum of seven distinct powers of γ, as shown here:

$$7 = \gamma^{-9} + \gamma^{-8} + \gamma^{-7} + \gamma^{-5} + \gamma^{-3} + \gamma^{-1} + \gamma^3.$$