Highest power of two dividing an entry of a matrix

1151. *Proposed by Gregory Dresden, Washington and Lee University, Lexington, VA.*

Fix an odd integer b in set $M = \begin{pmatrix} 1 & b \\ 4 & 5 \end{pmatrix}$. For a positive integer n, let $e(n)$ denote the exponent of the highest power of 2 that divides an entry of M^n. In other words, $2^{e(n)}$ divides some entry in M^n, but no larger power of 2 divides an entry of M^n. Find $e(n)$ as a function of n.

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.

The answer is $e(n) = 2 + \nu_2(n)$ where as usual we will use $\nu_2(k)$ to denote the exponent of the highest power of 2 which divides k. The characteristic polynomial of M is $P(X) = X^2 - 6X + 5 - 4b$. So, from the equality $M^2 = 6M + (4b - 5)I_2$ we deduce that $M^n = 6M^{n-1} + (4b - 5)M^{n-2}$ for all $n \geq 2$. Thus, if we write $M^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$, then all four sequences $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$, $(c_n)_{n\geq 0}$, and $(d_n)_{n\geq 0}$ satisfy the same recurrence relation (namely $x_n = 6x_{n-1} + (4b - 5)x_{n-2}$,) but they differ in their initial conditions:

$$(a_0, a_1) = (1, 1), \quad (b_0, b_1) = (0, b), \quad (c_0, c_1) = (0, 4), \quad (b_0, b_1) = (1, 5).$$

- The sequence $(a_n)_{n\geq 0}$ satisfies the recurrence $a_n \equiv a_{n-2} \mod 2$ and because $a_0 = a_1 = 1$ we see that a_n is odd for every n. The same argument shows that d_n is odd for every n.
- Let $(\delta_n)_{n\geq 0}$ the sequence defined recursively by

$$\delta_0 = 0,\ \delta_1 = 1, \quad \text{and} \quad \delta_n = 6\delta_{n-1} + (4b - 5)\delta_{n-2} \quad \text{for } n \geq 2.$$

  Then a simple induction shows that $b_n = b\delta_n$ and $c_n = 4\delta_n$ for all n. Since b is odd we see that $\nu_2(b_n) = \nu_2(\delta_n)$ while $\nu_2(c_n) = 2 + \nu_2(\delta_n)$ and $\nu_2(a_n) = \nu_(d_n) = 0$. We conclude that

$$e(n) = 2 + \nu_2(\delta_n).$$

- Let $\ell = (b + 1)/2 \in \mathbb{Z}$, and suppose that $\ell \neq 0$. We define $\alpha = 3 + 2\sqrt{2\ell}$ and $\beta = 3 - 2\sqrt{2\ell}$, the two zeros of the second degree trinomial $X^2 - 6X + 5 - 4b = 0$. Then

$$\begin{aligned} \delta_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} &= \frac{1}{4\sqrt{2\ell}} \sum_{k=0}^{n} \binom{n}{k} 3^{n-k} (2\sqrt{2\ell})^k (1 - (-1)^k) \\ &= \sum_{0\leq k<n/2}^{n} \binom{n}{2k+1} 3^{n-2k-1} (8l\ell)^k \equiv n3^{n-1} \mod 8. \end{aligned}$$

  In particular, this proves that if $n = 2m + 1$ (i.e., n is odd), then $\nu_2(\delta_n) = 0$. On the other hand,

$$\frac{\alpha^n + \beta^n}{2} = \sum_{k=0}^{n} \binom{n}{k} 3^{n-k}(2\sqrt{2\ell})^k \frac{1+(-1)^k}{2}$$
$$= \sum_{0\le k\le n/2}^{n} \binom{n}{2k} 3^{n-2k}(8\ell)^k = 1 \mod 2.$$

The equality $\delta_{2n} = (\alpha^n + \beta^n)\delta_n$ thus implies that $\nu_2(\delta_{2n}) = 1 + \nu_2(\delta_n)$. This shows inductively that $\nu_2(\delta_{2^k(2m+1)}) = k + \nu_2(\delta_{2m+1}) = k$; that is, $\nu_2(\delta_n) = \nu_2(n)$.

- It remains to consider the case $b = -1$. In this case $\delta_n = n3^{n-1}$, and the fact that $\nu_2(\delta_n) = \nu_2(n)$ for all $n \ge 1$ is immediate in this case.

Combining the above results we see that $e(n) = 2 + \nu_2(n)$, as announced.

Also solved by Armstrong Problem Solvers; Levent Batakci, Case Western Reserve U.; Ali Deeb and Hafez Al-Assad (jointly), Higher Inst. for Applied Sciences and Technology, Syria; Brendan Dosch (student), North Central C.; James Duemmel, Bellingham, WA; Florida Atlantic U. Problem Solving Group; Neville Fogarty and Chris Kennedy (jointly), Christopher Newport U.; George Washington U. Problems Group; Eugene Herman, Grinnell C.; John Kieffer, U. of Minnesota Twin Cities; Koopa Tak Lun Koo, Chinese STEAM Academy, Hong Kong; A. Bathi Kasturiarachi, Kent St. U. at Stark; Carl Libis, Columbia Southern U.; Albert Natian, Los Angeles Valley C.; Northwestern U. Math Problem Solving Group; Éric Pité, Paris, France; Francisco Perdomo and Ángel Plaza (jointly), Universidad de Las Palmas de Gran Canaria, Spain; Arthur Rosenthal, Salem St. U.; Ioannis Sfikas, Athens, Greece; Jacob Siehler, Gustavus Adolphus C.; Enrique Treviño, Lake Forest C.; Edward White, Frostburg, MD; and the proposer.

An inequality for the area of a triangle

1152. *Proposed by Yagoub Aliev, ADA University, Baku, Azerbaijan.*

Let R be the radius of the circumscribed circle of triangle ABC. Let D be a point on the arc BC which does not contain A, and drop perpendicular DE to BC. Now take point F on the same arc such that $\angle CAF = 2\angle BAF$. Prove that $8R \cdot \text{Area}(CDE) \le CF^3$.

Solution by Michel Bataille, Rouen, France.