1213. *Proposed by Rafael Jakimczuk, Universidad National de Lujá, Buenos Aires, Argentina.*

Let (a_n) be a sequence of positive integers, and for every positive integer n, define $P_n := (1 + \frac{1}{a_1 n})^{a_1} \cdot (1 + \frac{1}{a_2 n})^{a_2} \cdots (1 + \frac{1}{a_n n})^{a_n}$. Find $\lim_{n\to\infty} P_n$.

1214. *Proposed by Luis Moreno, SUNY Broome Community College, Binghampton, NY.*

The following sequence can be found in the text *Intermediate Analysis* by John Olmsted: $(1, 2, 2\frac{1}{2}, 3, 3\frac{1}{3}, 3\frac{2}{3}, 4, 4\frac{1}{4}, 4\frac{2}{4}, 4\frac{3}{4}, 5, \ldots)$. Now let n be a positive integer. Find a closed-form expression for a_n, the nth term of the above sequence.

1215. *Proposed by Greg Oman, University of Colorado at Colorado Springs, Colorado Springs, CO.*

Let R be a ring (assumed only to be associative but not to contain an identity unless stated). Recall that a *subring* of R is a nonempty subset of R closed under addition, negatives, and multiplication. Find all rings R with identity $1 \neq 0$ with the property that no proper, nontrivial subring of R has an identity (which need NOT be the identity of R).

SOLUTIONS

A continued fraction given by Fibonacci

1186. *Proposed by Gregory Dresden, Washington and Lee University, Lexington, VA and ZhenShu Luan (high school student), St. George's School, Vancouver, BC, Canada.*

Find a closed-form expression for the continued fraction $[1, 1, \ldots, 1, 3, 1, 1, \ldots, 1]$, which has n ones before, and after, the middle three.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

In order to get the desired expression, we recall the following elegant way of evaluating the convergents of a continued fraction. [See, for instance, https://de.wikipedia.org/wiki/Kettenbruch, particularly the paragraph "matrixdarstellung."] We have to evaluate the product

$$\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}^n \cdot \begin{bmatrix}3 & 1\\ 1 & 0\end{bmatrix} \cdot \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}^n$$

Let F_n be the nth Fibonacci number. From the familiar representation

$$[1, 1, ..., 1] = \frac{F_{n+1}}{F_n},$$

(with n 1's), we get

$$\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}^n = \begin{bmatrix}F_{n+1} & F_n\\ F_n & F_{n-1}\end{bmatrix},$$

whence

$$\begin{bmatrix}1 & \\ 1 & 0\end{bmatrix}^n \cdot \begin{bmatrix}3 & 1\\ 1 & 0\end{bmatrix} \cdot \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}^n = \begin{bmatrix}F_{n+1} & F_n\\ F_n & F_{n-1}\end{bmatrix} \cdot \begin{bmatrix}3 & 1\\ 1 & 0\end{bmatrix} \cdot \begin{bmatrix}F_{n+1} & F_n\\ F_n & F_{n-1}\end{bmatrix};$$

that is

$$\begin{bmatrix} 1 & \\ 1 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$$

$$= \begin{bmatrix} F_{n+1} \cdot (3F_{n+1} + 2F_n) & F_{n+1} \cdot F_{n-1} + F_n\,(3F_{n+1} + F_n) \\ F_{n+1} \cdot F_{n-1} + 3F_n F_{n+1} + F_n^2 & F_n\,(2F_{n-1} + 3F_n) \end{bmatrix}.$$

This leads to the desired closed-form expression of [1, ..., 1, 3, 1, ..., 1]:

$$\begin{aligned} \frac{F_{n+1}\,(3F_{n+1} + 2F_n)}{F_{n+1} \cdot F_{n-1} + 3F_n \cdot F_{n+1} + F_n^2} &= \frac{F_{n+1}\,(3F_{n+1} + 2F_n)}{F_{n+1}\,(F_{n+1} - F_n) + F_n\,(3F - n + 1 + F_n)} \\ &= \frac{F_{n+1}\,(3F_{n+1} + 2F_n)}{F_{n+1}^2 + 2F_{n+1} \cdot F_n + F_n^2} \\ &= \frac{F_{n+1}\,(3F_{n+1} + 2F_n)}{(F_{n+1} + F_n)^2} \\ &= \frac{F_{n+1}\,(3F_{n+1} + 2F_n)}{F_{n+2}^2} \\ &= \frac{F_{n+1}\,(F_{n+1} + 2F_{n+2})}{F_{n+2}^2}. \end{aligned}$$

This and

$$F_{n+1} + 2F_{n+2} = F_{n+3} + F_{n+2} = F_{n+4}$$

yield the closed-form result

$$\frac{F_{n+1}F_{n+4}}{F_{n+2}^2}.$$

Also solved by Brian Beasley, Presbyterian C.; Anthony Bevelacqua, U. of N. Dakota; Brian Bradie, Christopher Newport U.; James Brenneis, Penn State - Shenango; Hongwei Chen, Christopher Newport U.;Giuseppe Fera, Vicenza, Italy; Eugene Herman, Grinnell C.; Donald Hooley, Bluffton, OH; Joel Iiams, U. of N. Dakota; Harris Kwong, SUNY Fredonia; Seungheon Lee, Yonsei U.; Carl Libis, Columbia Southern U.; Graham Lord, Princeton, NJ; Ioana Mihaila, Cal Poly Pomona; Missouri State U. Problem Solving Group; Northwestern U. Math Problem Solving Group; Randy Schwartz, Schoolcraft C. (retired); Albert Stadler, Herrliberg, Switzerland; Paul Stockmeyer, C. of William and Mary; David Terr, Oceanside, CA; Enrique Treviño, Lakeforest C.; Michael Vowe, Therwil, Switzerland; and the proposer.

A limit of maxima

1187. *Proposed by Reza Farhadian, Lorestan University, Khorramabad, Iran.*

Let $\alpha > 1$ be a fixed real number, and consider the function $M\colon [1, \infty) \to \mathbb{N}$ defined by $M(x) = \max\{m \in \mathbb{N}\colon m! \le \alpha^x\}$. Prove the following:

$$\lim_{n\to\infty} \frac{\sqrt[n]{M(1)M(2)\cdots M(n)}}{M(n)} = e^{-1}.$$