

 (x, r, θ) is $h(x, r, \theta) = (1/\pi) f(x) g(r) = (4r/\pi^2) (1 - x^2)^{-1/2}$ on $[0, 1] \times [0, 1] \times [0, \pi]$. The area of $\triangle ABC$ is $[ABC] = |x - r \cos \theta| (1 - x^2)^{1/2}$ (see the figure above). With D chosen uniformly at random, the probability of the event \mathcal{E} that D lies within $\triangle ABC$ is

$$\mathbb{P}[\mathcal{E} \mid A, B, C] = \frac{[ABC]}{\pi} = \frac{1}{\pi} |r \cos \theta - x| (1 - x^2)^{1/2}.$$

Thus, the desired probability is given by

$$\int_{0}^{1} \int_{0}^{1} \int_{0}^{\pi} \mathbb{P}[\mathcal{E} \mid A, B, C] \cdot h(x, r, \theta) \, d\theta \, dr \, dx = \int_{0}^{1} \int_{0}^{1} \int_{0}^{\pi} \frac{4r}{\pi^{3}} |r \cos \theta - x| \, d\theta \, dr \, dx$$
$$= \frac{5}{4\pi^{2}} \approx 12.7\%.$$

Also solved by Randy K. Schwartz, Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Jan A. Grzesik, Robert Calcaterra, Peter McPolin (UK), and the proposer.

Algebraic numbers with common decimal tails

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2035. Proposed by Gregory Dresden, Prakriti Panthi (student), Anukriti Shrestha (student) and Jiahao Zhang (student), Washington & Lee University, Lexington, VA.

Two real numbers x, y are said to have a common decimal part if xy < 0 and x + y is an integer, or else $xy \ge 0$ and x - y is an integer. More concretely, this means that the decimal expansions of x, y are of the forms

$$\pm a_m a_{m-1} \dots a_1 a_0 \cdot d_1 d_2 d_3 \dots,$$

 $\pm b_n b_{n-1} \dots b_1 b_0 \cdot d_1 d_2 d_3 \dots,$

where the common decimal part is $0.d_1d_2d_3...$

Find all polynomials of degree at least 2 with integer coefficients, all roots real, and irreducible over the rationals, whose roots have pairwise common decimal tails.

Solution by Jacob Siehler, Gustavus Adolphus College, Saint Peter, MN.

We show that the property in the statement of the problem characterizes those polynomials that are quadratic with integer coefficients and irreducible over \mathbb{Q} such that ac < 0 and a|b. (The irreducibility amounts to the discriminant $b^2 - 4ac$ not being a perfect square.)

Lemma 1. A polynomial with integer coefficients, irreducible over \mathbb{Q} , cannot have two roots that differ by an integer.

Proof. Suppose f were such a polynomial, with two roots α and $\alpha + c$, where c is a nonzero integer. By irreducibility of f, the Galois group of the splitting field K of f over $\mathbb Q$ acts transitively on the roots of f, so there is a field automorphism ϕ of K with $\phi(\alpha) = \alpha + c$. Since c is rational and hence fixed by ϕ , it follows by induction that $\alpha, \alpha + c, \alpha + 2c, \ldots, \alpha + nc, \ldots$ are distinct roots of f (each successive one the image under ϕ of the previous), contradicting the fact that a polynomial has only finitely many roots.

Lemma 1 implies that a polynomial f meeting the requirements of the problem cannot have two roots of the same sign. Therefore, any such f must necessarily have exactly two distinct roots of opposite signs. Since f is irreducible over $\mathbb Q$ and nonlinear, it has no repeated roots and no zero root. Thus, f must be quadratic and its two roots must be real of opposite signs with common decimal tails (i.e., integer sum). The following lemma characterizing these quadratics in terms of their coefficients concludes our solution.

Lemma 2. Let $f(x) = ax^2 + bx + c$ be irreducible over \mathbb{Q} , with integer coefficients (where $a \neq 0$). The roots of f are real of opposite sign and have common decimal tails if and only if ac < 0 and a|b.

Proof. Since f is quadratic and irreducible over \mathbb{Q} , it has exactly two distinct roots α and β .

If α and β are opposite-sign reals with common decimal tails, then $-b/a = \alpha + \beta$ is an integer, and furthermore $c/a = \alpha\beta < 0$, hence ac < 0 also.

Conversely, if ac < 0 and a|b, then $b^2 - 4ac > 0$, so α and β are real numbers; furthermore, they have opposite signs since $\alpha\beta = c/a < 0$ (as ac < 0 by assumption). By the assumption a|b, it follows that $\alpha + \beta = -b/a$ is an integer, so α and β have common decimal tails.

Also solved by Robert Calcaterra, Michael Reid, and the proposer.

Answers

Solutions to the Quickies from page 389.

A1085. For all positive x, y, z we have the well-known inequality

$$x^{3} + y^{3} + z^{3} + 3xyz \ge x^{2}y + x^{2}z + xy^{2} + y^{2}z + xz^{2} + yz^{2},$$

which is the special case t = 1 of Schur's inequality $x^t(x - y)(x - z) + y^t(y - z)(y - x) + z^t(z - x)(z - y) \ge 0$ valid for nonnegative x, y, z. (Wikipedia page: https://en. wikipedia.org/wiki/Schur's_inequality.) Taking $x = a^n$, $y = b^n$ and $z = c^n$ above for $n = 0, 1, 2, \ldots$, and adding the resulting inequalities using the geometric formula,