

Lemma 1. A polynomial with integer coefficients, irreducible over \mathbb{Q} , cannot have two roots that differ by an integer.

Proof. Suppose f were such a polynomial, with two roots α and $\alpha + c$, where c is a nonzero integer. By irreducibility of f , the Galois group of the splitting field K of f over \mathbb{Q} acts transitively on the roots of f , so there is a field automorphism ϕ of K with $\phi(\alpha) = \alpha + c$. Since c is rational and hence fixed by ϕ , it follows by induction that $\alpha, \alpha + c, \alpha + 2c, \dots, \alpha + nc, \dots$ are distinct roots of f (each successive one the image under ϕ of the previous), contradicting the fact that a polynomial has only finitely many roots. ■

Lemma 1 implies that a polynomial f meeting the requirements of the problem cannot have two roots of the same sign. Therefore, any such f must necessarily have exactly two distinct roots of opposite signs. Since f is irreducible over \mathbb{Q} and non-linear, it has no repeated roots and no zero root. Thus, f must be quadratic and its two roots must be real of opposite signs with common decimal tails (i.e., integer sum). The following lemma characterizing these quadratics in terms of their coefficients concludes our solution.

Lemma 2. Let $f(x) = ax^2 + bx + c$ be irreducible over \mathbb{Q} , with integer coefficients (where $a \neq 0$). The roots of f are real of opposite sign and have common decimal tails if and only if $ac < 0$ and $a|b$.

Proof. Since f is quadratic and irreducible over \mathbb{Q} , it has exactly two distinct roots α and β .

If α and β are opposite-sign reals with common decimal tails, then $-b/a = \alpha + \beta$ is an integer, and furthermore $c/a = \alpha\beta < 0$, hence $ac < 0$ also.

Conversely, if $ac < 0$ and $a|b$, then $b^2 - 4ac > 0$, so α and β are real numbers; furthermore, they have opposite signs since $\alpha\beta = c/a < 0$ (as $ac < 0$ by assumption). By the assumption $a|b$, it follows that $\alpha + \beta = -b/a$ is an integer, so α and β have common decimal tails. ■

Also solved by Robert Calcaterra, Michael Reid, and the proposer.

Answers

Solutions to the Quickies from page 389.

A1085. For all positive x, y, z we have the well-known inequality

$$x^3 + y^3 + z^3 + 3xyz \geq x^2y + x^2z + xy^2 + y^2z + xz^2 + yz^2,$$

which is the special case $t = 1$ of Schur's inequality $x^t(x - y)(x - z) + y^t(y - z)(y - x) + z^t(z - x)(z - y) \geq 0$ valid for nonnegative x, y, z . (Wikipedia page: https://en.wikipedia.org/wiki/Schur's_inequality.) Taking $x = a^n$, $y = b^n$ and $z = c^n$ above for $n = 0, 1, 2, \dots$, and adding the resulting inequalities using the geometric formula,