Note that

$$b_n - b_{n-1} = \frac{H_{n-1}\sum_{i=1}^{n}\frac{x_i}{i} - H_n\sum_{i=1}^{n-1}\frac{x_i}{i}}{H_{n-1}H_n} = \frac{\frac{x_n}{n}\sum_{i=1}^{n-1}\frac{1}{i} - \frac{1}{n}\sum_{i=1}^{n-1}\frac{x_i}{i}}{H_{n-1}H_n}.$$

Since (x_n) takes only the values 0, 1, the equation above shows that $b_n \geq b_{n-1}$ if $x_n = 1$, while $b_n \leq b_{n-1}$ if $x_n = 0$. It follows that

$$\limsup_{k\to\infty} b_k = \lim_{n\to\infty} b_{(2n+1)!-1} = \frac{1}{2}.$$

A similar argument shows that $\liminf_{k\to\infty} b_k = \lim_{n\to\infty} b_{(2n)!-1} = 1/2$; this shows that $\lim_{n\to\infty} b_n = 1/2$.

Also solved by Robert Calcaterra, Dmitry Fleischman, Russell Gordon, Eugene A. Herman, Elias Lampakis (Greece), José Heber Nieto (Venezuela), and the proposer. There was one incomplete or incorrect solution.

A sextic with Galois group S_3 **December 2018**

2058. *Proposed by Gregory Dresden, Saimon Islam (student) and Jiahao Zhang (student), Washington & Lee University, Lexington, VA.*

Let a be a rational number such that the polynomial

$$f(x) = x^6 + 3x^5 - ax^4 - (2a+5)x^3 - ax^2 + 3x + 1$$

is irreducible over $\mathbb{Q}$, and let F be the splitting field for $f(x)$ over $\mathbb{Q}$. Find the Galois group $\mathrm{Gal}(F/\mathbb{Q})$ (up to isomorphism).

Solution by Robert Calcaterra, University of Wisconsin-Platteville, Platteville, WI.
The Galois group $G = \mathrm{Gal}(F/\mathbb{Q})$ is isomorphic to the symmetric group S_3. Observe that $f(x)$ is palindromic, so $x^6 f(1/x) = f(x)$; also, $f(-x-1) = f(x)$. It follows that z is a zero of $f(x)$ if and only if $\iota(z) := 1/z$ is a zero thereof, if and only if $\tau(z) := -z-1$ is. Thus, ι and τ are involutions (i.e., ι^2 and τ^2 are both the identity transformation) acting on the set of zeros of f. (They may be regarded formally as projective transformations of $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.) Let $\sigma = \iota\tau$ be the transformation $z \mapsto -1/(1+z)$. The group of transformations generated by ι and τ is evidently the same as that generated by σ and τ (since $\sigma = \iota\tau$ and $\iota = \sigma\tau$). It is easy to check that σ^3 is the identity transformation, and $\tau\sigma = \sigma^2\tau$. Therefore, the group $\mathfrak{T} = \langle\iota, \tau\rangle = \langle\sigma, \tau\rangle$ generated by ι, τ (or by σ, τ) is isomorphic to the dihedral group D_6 (i.e., to the symmetric group S_3); it consists of the elements id, $\sigma, \sigma^2, \tau, \sigma\tau, \sigma^2\tau$. Since ι, τ act on the set of roots of $f(x)$, so does $\mathfrak{T}$.

Lemma. *$\mathfrak{T}$ acts on the set of roots of $f(x)$ simply, i.e., given a root z of $f(x)$ and transformations $\alpha \neq \beta$ in $\mathfrak{T}$, we have $\alpha(z) \neq \beta(z)$.*

Proof. Note that the group $\mathfrak{T}$ consists of degree-1 projective transformations with coefficients in $\mathbb{Q}$, i.e., $\alpha(z)$ and $\beta(z)$ are quotients of polynomials of degree at most 1 in z, not both constant, with coefficients in $\mathbb{Q}$. We see that $\alpha(z)$ and $\beta(z)$ are roots of $f(x)$ since $\mathfrak{T}$ acts on roots of $f(x)$ and z is one such root. If we had $\alpha(z) = \beta(z)$, clearing denominators in this equation one sees that z would be root of a linear or quadratic equation with rational coefficients, contradicting the hypothesis that z is a root of the degree-6 polynomial $f(x)$ that is irreducible over $\mathbb{Q}$. ■

Fix a root z of $f(x)$. By the lemma above, the set of six distinct roots of $f(x)$ is $\{\alpha(z) : \alpha \in \mathfrak{T}\}$. For all $\alpha \in \mathfrak{T}$, the complex number $\alpha(z)$ is a rational expression in z

with rational coefficients; thus, every field automorphism $g \in G$ satisfies $g(\alpha(z)) = \alpha(g(z))$. Since $f(x)$ is irreducible, G acts transitively on the set of these six roots; in particular, for each $\alpha \in \mathfrak{T}$ there is $g \in G$ such that $\alpha(z) = g(z)$. If $g, h \in G$ satisfy $g(z) = \alpha(z) = h(z)$, then the images under g, h of any fixed root z' of $f(x)$ (necessarily of the form $z' = \beta(z)$ for some $\beta \in \mathfrak{T}$) must coincide: $g(z') = g(\beta(z)) = \beta(g(z)) = \beta\alpha(z) = \beta(h(z)) = h(\beta(z)) = h(z')$. It follows that, given $\alpha \in \mathfrak{T}$, a unique automorphism $g = g_\alpha$ of F is determined by the condition $g(z) = \alpha(z)$. Given $\alpha, \beta \in \mathfrak{T}$, we have $g_{\alpha\beta} = g_\beta g_\alpha$, since $g_{\alpha\beta}(z) = \alpha\beta(z) = \alpha(g_\beta(z)) = g_\beta(\alpha(z)) = g_\beta(g_\alpha(z))$. Therefore, $\alpha \mapsto g_\alpha$ is an isomorphism between G and the opposite group of $\mathfrak{T}$, which is still isomorphic to S_3.

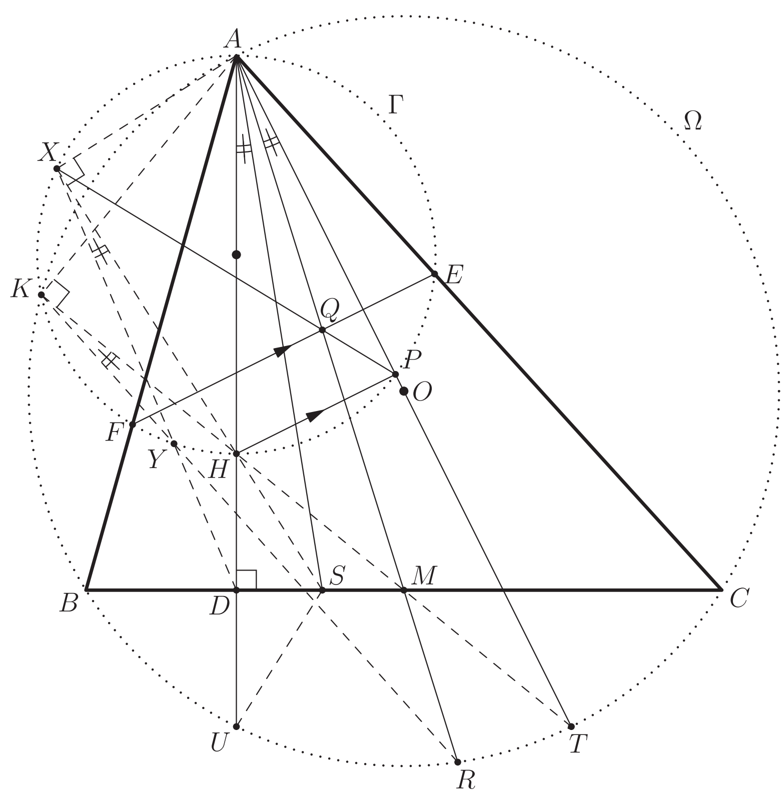
Also solved by Anthony Bevelacqua, Peter McPolin (Northern Ireland), Michael Reid, and the proposer.

A canonical similarity transformation of a given triangle **December 2018**

2059. *Proposed by Andrew Wu, St. Albans School, McLean, VA.*

Let triangle $\triangle ABC$ be acute and scalene with orthocenter H, altitudes $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$, and circumcircle Ω. Let Γ be the circle with diameter $\overline{AH}$. Circles Γ and Ω intersect at A and at a second point K. Let point P lie on Γ so that $\overline{HP}$ is parallel to $\overline{EF}$. Let M be the midpoint of $\overline{BC}$. Let $\overleftrightarrow{AM}$ intersect Ω at $R \neq A$, and $\overline{EF}$ at Q. Let $\overleftrightarrow{PQ}$ meet Γ again at $X \neq P$. Show that $\overline{DX}$ and $\overline{KR}$ concur on Γ.

Solution by Kyle Gatesman (student), Johns Hopkins University, Baltimore, MD.



Let Y be the intersection of $\overline{DX}$ and $\overline{KR}$. Let U be the reflection of H on $\overleftrightarrow{BC}$; thus, $\angle USD = \angle HSD$, and it is well known that U lies on Ω. Let O be the circumcenter of