Proposed Problem.

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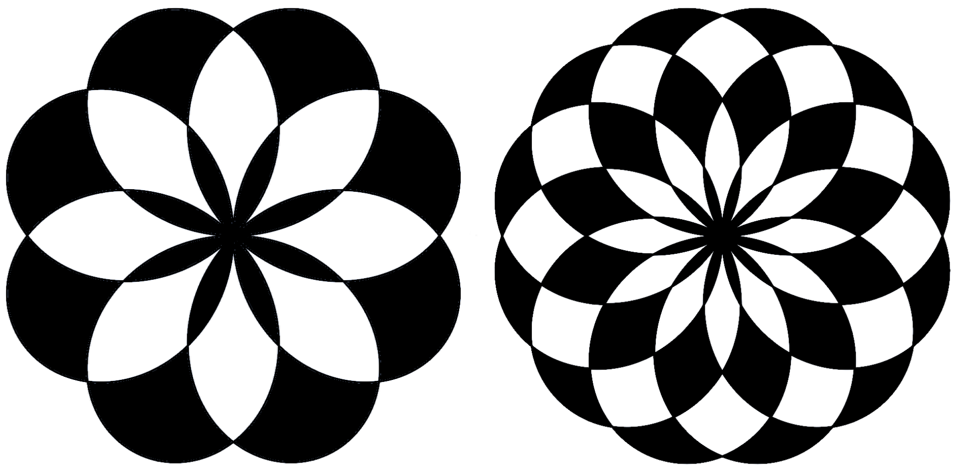
PROBLEM:

Shown here are the graphs of $r = \sin 4\theta/3$ and $r = \sin 6\theta/5$, where every other adjacent region (starting from the outside) is shaded black.



Find the total shaded area (as a function of k) for any such graph $r = \sin(k+1)\theta/k$, where $k > 0$ is an odd integer and θ ranges from 0 to $2k\pi$.

SOLUTION, by John Chapman, Yvonne Cheng, and Gregory Dresden.

The area, for k odd, is always $\pi/2$.

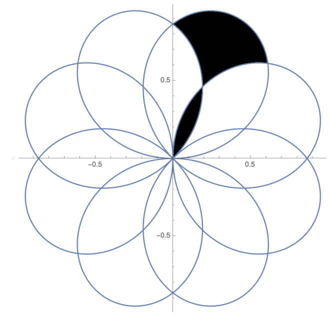
We can also show that for k even, then the area is $\left(k \tan \dfrac{\pi}{2k}\right)$, which rapidly approaches $\pi/2$ from above as $k \to \infty$.

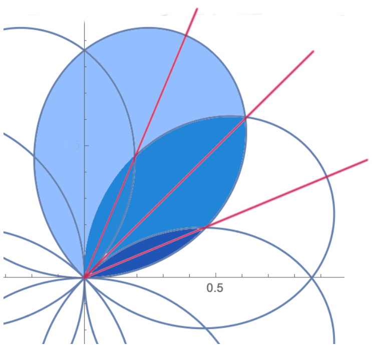
Here's the proof. (see next page)

First, let's assume that k is odd.

We will show that the shaded area for $r = \sin 4\theta/3$ (the first picture in the statement of the problem) is $\pi/2$. We then explain how an identical technique will work for $r = \sin 6\theta/5$ (the second picture) and likewise for any $r = \sin(k+1)\theta/k$ for k odd.

To begin with, we break down the picture for $r = \sin 4\theta/3$ by looking at the area of just one "ginkgo leaf and stem", as shown on the right. (The full picture is made up of eight of these ginkgo leaves and stems.)





Now, to understand the area of this ginkgo leaf and stem, we consider this picture on the left. Notice that we have shaded in a large petal, a medium petal (which overlaps the large petal), and a small petal (which overlaps the medium and large). By appealing to symmetry, and by looking at the three lines in the picture on the right (the lines are located at $\pi/8$, $2\pi/8$, and $3\pi/8$), we conclude that area of the large petal (we denote this area by A_3) is

$$A_3 = 2 \cdot \int_0^{3\pi/8} \frac{1}{2} r^2 \, d\theta, \qquad \text{... where } r = \sin 4\theta/3,$$

and likewise the areas of the medium and small petals (call them A_2 and A_1) are given by similar integrals but with the upper limit of $3\pi/8$ replaced by $2\pi/8$ and $\pi/8$, respectively. A bit of computation gives the following values for the areas A_3, A_2, and A_1.

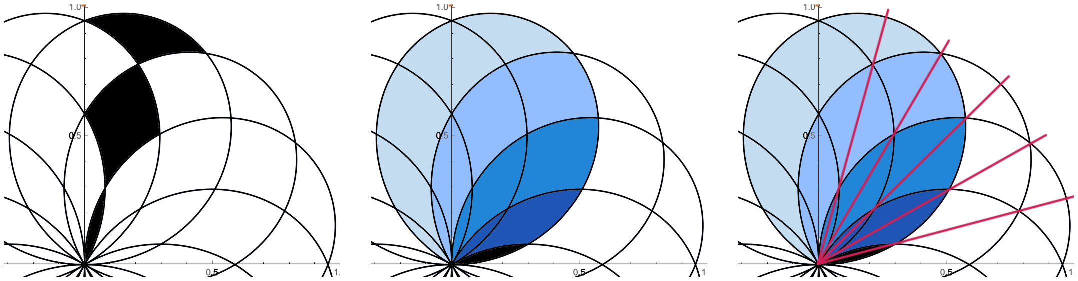
$$\begin{aligned} A_3 &= (3\pi - 3\sin 3\pi/3)/16 = (3\pi)/16, \\ A_2 &= (2\pi - 3\sin 2\pi/3)/16, \\ A_1 &= (1\pi - 3\sin 1\pi/3)/16. \end{aligned}$$

Looking once more at our ginkgo leaf and stem, we see that the area of the ginkgo leaf can be found by starting with a large petal, subtracting two medium petals, and then compensating for the overlap in the medium petals by adding back a small petal. In other words, the area of the ginkgo "leaf" is $A_3 - 2A_2 + A_1$. As for the ginkgo "stem", its area is the area of a small petal, A_1.
Summing up, the area of the ginkgo leaf and stem is $A_3 - 2A_2 + 2A_1$, and this is

$$A_3 - 2A_2 + 2A_1 \;=\; (3 - 2(2-1))\pi/16 \;+\; 6(\sin 2\pi/3 - \sin 1\pi/3)/16 \;=\; \pi/16,$$

and that last step is because $\sin x = \sin(\pi - x)$. Finally, since there are eight ginkgo leaves in our full picture, the total area is $\pi/2$.

Moving on to the next case, we will apply a similar technique and we will see a pattern develop. Consider the following pictures of the graph of $r = \sin 6\theta/5$ in the first quadrant. Note that what was a "ginkgo leaf and stem" from the last example now has three parts instead of two; we will simply call this a "ginkgo assembly".



We will denote the areas of the five petals (from the middle picture) by B_5 for the largest petal, down to B_1 for the smallest, and each of these can be calculated by again using symmetry and looking at the red lines on the left (each separated by an angle of $\pi/12$). We obtain the following values:

$$\begin{aligned}
B_5 &= (5\pi - 5\sin 5\pi/5)/24 = (5\pi)/24,\\
B_4 &= (4\pi - 5\sin 4\pi/5)/24,\\
B_3 &= (3\pi - 5\sin 3\pi/5)/24,\\
B_2 &= (2\pi - 5\sin 2\pi/5)/24,\\
B_1 &= (1\pi - 5\sin 1\pi/5)/24.
\end{aligned}$$

Looking once more at our "ginkgo assembly", we see that the area of the outer blade of the assembly is $B_5 - 2B_4 + B_3$, and the area of the middle blade is $B_3 - 2B_2 + B_1$, and of course the smallest part has area B_1. Summing up, we have $B_5 - 2B_4 + 2B_3 - 2B_2 + 2B_1$, and this is

$$(5 - 2(4 - 3 + 2 - 1))\pi/24 + 10(\sin 4\pi/5 - \sin 3\pi/5 + \sin 2\pi/5 - \sin 1\pi/5)/24 \;=\; \pi/24,$$

and that last equality is again because $\sin x = \sin(\pi - x)$. Since there are twelve such ginkgo assemblies, the total area is again $\pi/2$.

With these two examples in hand, it is no surprise that in general, for $r = \sin(k+1)\theta/k$ with $k > 5$ odd, there are $2(k+1)$ of these "ginkgo assemblies". The area of each such ginko assembly is found by the following sum:

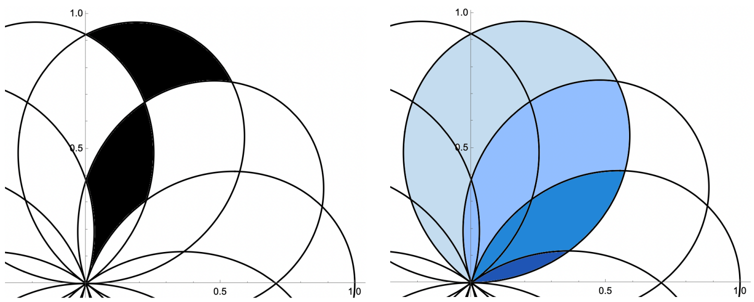
$$\begin{aligned}
&(k - 2((k-1) - (k-2) + (k-3) - (k-4) + \cdots + 2 - 1))\pi/4(k+1)\\
&\qquad + 2k(\sin(k-1)\pi/k - \sin(k-2)\pi/k + \cdots + \sin 2\pi/k - \sin 1\pi/k)/4(k+1),
\end{aligned}$$

and this simplifies to $\pi/4(k+1)$. When we multiply by the number of ginkgo assemblies, we obtain a total area of $\pi/2$, as desired.

Finally, we note that for $k = 1$, we get the familiar four-petal rose $r = \sin 2\theta$ and it is easy to show that the total area enclosed by each outer petal (and all the petals are outer petals) is again $\pi/2$.

Now, let's assume that k is even.

Shown here are two pictures of the graph $r = \sin 5\theta/4$ in the first quadrant.
Note that we still have the "ginkgo leaf and stem" as seen in the previous graph of $r = \sin 4\theta/3$.



We will denote the areas of the four petals (from the second picture) by C_4 for the largest (light blue) petal, down to C_1 for the smallest (dark blue), and each of these can be calculated by once again using symmetry and by modeling our previous work. We obtains the following values:

$$
\begin{aligned}
C_4 &= (4\pi - 4\sin 4\pi/4)/20 \;\;=\;\; (4\pi)/20, \\
C_3 &= (3\pi - 4\sin 3\pi/4)/20, \\
C_2 &= (2\pi - 4\sin 2\pi/4)/20, \\
C_1 &= (1\pi - 4\sin 1\pi/4)/20.
\end{aligned}
$$

Looking once more at our ginkgo leaf and stem, we see that the area of the outer leaf is $C_4 - 2C_3 + C_2$, and the area of the stem is $C_2 - 2C_1$. Summing up, we have $C_4 - 2C_3 + 2C_2 - 2C_1$, and this is

$$(4 - 2(3 - 2 + 1))\pi/20 + 8(\sin 3\pi/4 - \sin 2\pi/4 + \sin \pi/4)/20,$$

and this time it is the first terms that cancel while the sines in the second expression do not. In fact, a direct calculation shows that the above expression is equal to

$$8(\sin 3\pi/4 - \sin 2\pi/4 + \sin \pi/4)/20 \;\;=\;\; (4/10)\tan \pi/8,$$

and since there are ten such ginkgo assemblies, the total area is $4\tan \pi/8$.

Let us now jump to the general case for $r = \sin(k+1)\theta/k$ with $k \geq 4$ even. There will be $2(k+1)$ of these "ginkgo assemblies". The area of each such ginko assembly is found by the following sum:

$$
\begin{aligned}
&(k - 2((k-1) - (k-2) + (k-3) - (k-4) + \cdots - 2 + 1))\pi/4(k+1) \\
&\qquad + 2k(\sin(k-1)\pi/k - \sin(k-2)\pi/k + \cdots + - \sin 2\pi/k + \sin 1\pi/k)/4(k+1),
\end{aligned}
$$

and this simplifies to the following sum:

$$\frac{2k}{4(k+1)}\left(\sin 1\pi/k - \sin 2\pi/k + \cdots + \sin(k-1)\pi/k\right).$$

Thanks to a trig identity I haven't quite figured out, this reduces to

$$\frac{2k}{4(k+1)}\left(\tan\frac{\pi}{2k}\right),$$

and when we multiply this by $2(k+1)$ to account for the total number of ginkgo assemblies, this becomes

$$k\tan\frac{\pi}{2k},$$

and furthermore we can apply L'Hopital's to show that as $k \to \infty$, this converges to $\pi/2$.