$2x+1>0$. Thus, if $g(x, y)$ is (a positive) prime, we must have $2x-1=1$ so that $x=1$. Since $y^2=2x^2-1$, we see that $|y|=1$ and $g(x, y)=3$.

Also solved by REZA AKHLAGHI, Big Sandy Community and Technical C.; HERB BAILEY and JOHN RICKERT (jointly), Rose-Hulman Institute of Technology; MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian C.; DAVID BRESSOUD and STAN WAGON (jointly), Macalester C.; STAN BYRD, U. of Tennessee-Chattanooga; MINH CAN, Brooks C.; JOHN CHRISTOPHER, California State U.-Sacramento; PHIL CLARKE, Los Angeles Valley C.; ELLIOTT COHEN, Fontenay-sous-Bois, France; CON AMORE PROBLEM GROUP, The Danish U. of Education, Copenhagen, Denmark; CHIP CURTIS, Missouri Southern State C.; JIM DELANY, California Polytechnic State U.; HABIB Y. FAR, Montgomery C.; DMITRY FLEISCHMAN, Santa Monica, CA; MATT FOSS, North Hennepin C.C.; OVIDIU FURDUI (student), Western Michigan U.; G.R.A. 20 MATH PROBLEMS GROUP, Rome, Italy; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; PETER HOHLER, Aarburg, Switzerland; RICKY IKEDA, Leeward C.C.; KHUDIJA JAMIL (student), California State U.-Northridge; ALEXANDER KOONCE, U. of Redlands; KENNETH KORBIN, New York, NY; HARRIS KWONG, SUNY C. at Fredonia; NORTHWESTERN U. MATH PROBLEM SOLVING GROUP; PHILIP OPPENHEIMER, Norwalk, CT; MOHAMMAD RIAZI-KERMANI, Fort Hays State U.; SAINT ANSELM C. PROBLEM SOLVERS; WILLIAM SEAMAN, Albright C.; RICHARD M. SMITH; JOHN HENRY STEELMAN, Indiana U. of Pennsylvania; DAVID STOLP (student), California State U.-Chico; H. T. TANG, Hayward, CA; THOMAS WALES, Cambridge, MA; HONGBIAO ZENG, Fort Hays State U.; LI ZHOU, Polk C. C.; and the proposer.

Editors' notes: Solver John Christopher used the fact that primes congruent to 1 modulo 4 can be written as the sum of squares in (essentially) only one way to show that 5 is the only prime assumed by $f(x, y)=(2x^2)^2+(y^2)^2=(2xy)^2+(y^2-2x^2)^2$.

Solvers Michel Bataille, David Bressoud & Stan Wagon, Habib Y. Far, John Henry Steelman, and David Stolp considered negative values of the function $g(x, y)=4x^4-y^4$ and discovered the following "negative prime" values: $g(2,3)=-17$, $g(12,17)=-577$ and $g(408,577)=-665857$. Bressoud and Wagon show that these are the only negative prime values with fewer than 800,000 digits that are assumed by $g(x, y)$, and Far claims that these three values are the only negative prime values assumed by $g(x, y)$.

A critical parameter for a family of polar curves

758. *Proposed by Gregory Dresden, Washington & Lee University, Lexington, VA*

For b a real number, let $L(b)$ be the arc length of the polar graph $r=(1-b)+b\cos(\theta)$ with θ in the interval $[0,2\pi]$.

(a) Find the extreme values of the function L.

(b) Find all values b for which the function L is differentiable.

Solution by William Seaman, Albright College, Reading, PA

(a) We have

$$\begin{aligned}
L(b) &= \int_0^{2\pi}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta=\int_0^{2\pi}\sqrt{1-2b(1-b)(1-\cos(\theta))}\,d\theta \\
&= 2\int_0^{\pi}\sqrt{1-2b(1-b)(1-\cos(\theta))}\,d\theta = 2\int_0^{\pi}\sqrt{1-4b(1-b)\sin^2\left(\frac{\theta}{2}\right)}\,d\theta \\
&= 4\int_0^{\pi/2}\sqrt{1-4b(1-b)\sin^2(\theta)}\,d\theta = 4\int_0^{\pi/2}\sqrt{\cos^2(\theta)+(1-2b)^2\sin^2(\theta)}\,d\theta
\end{aligned}$$

from which it immediately follows that L has no maximum value and that its minimum value is $L(\frac{1}{2}) = 4$.

(b) For $b \neq \frac{1}{2}$, we conclude from the Leibniz rule (for interchanging differentiation and integration) that $L'(b)$ exists. We will now argue that $L'(\frac{1}{2})$ exists and is equal to 0. If we let

$$D(\theta) = \sqrt{\cos^2(\theta) + (1-2b)^2 \sin^2(\theta)} + \cos(\theta)$$

then elementary algebra shows that

$$\left| \frac{L(b) - L\left(\frac{1}{2}\right)}{b - \frac{1}{2}} \right| = \left| \frac{L(b) - 4}{b - \frac{1}{2}} \right| = 16 \left| b - \frac{1}{2} \right| \int_0^{\pi/2} \frac{\sin^2(\theta)}{D(\theta)}\, d\theta \tag{1}$$

Note that

$$\frac{\sin^2(\theta)}{D(\theta)} \leq \frac{1}{\cos(\theta)} \quad \text{for} \quad 0 \leq \theta < \frac{\pi}{2} \tag{2}$$

and that

$$\frac{\sin^2(\theta)}{D(\theta)} \leq \frac{1}{|1-2b|} \quad \text{for} \quad 0 \leq \theta \leq \frac{\pi}{2}. \tag{3}$$

Let $0 < \varepsilon < \frac{\pi}{2}$ be given. Then

$$\int_0^{\pi/2} \frac{\sin^2(\theta)}{D(\theta)}\, d\theta = \int_0^{(\pi/2)-\varepsilon} \frac{\sin^2(\theta)}{D(\theta)}\, d\theta + \int_{(\pi/2)-\varepsilon}^{\pi/2} \frac{\sin^2(\theta)}{D(\theta)}\, d\theta \tag{4}$$

and since $\cos(\theta)$ is decreasing on $[0, \frac{\pi}{2})$, it follows from inequality (2) that

$$\int_0^{(\pi/2)-\varepsilon} \frac{\sin^2(\theta)}{D(\theta)}\, d\theta \leq \frac{\frac{\pi}{2}}{\cos\left(\frac{\pi}{2} - \varepsilon\right)}. \tag{5}$$

From inequality (3) we have

$$\int_{(\pi/2)-\varepsilon}^{\pi/2} \frac{\sin^2(\theta)}{D(\theta)}\, d\theta \leq \frac{\varepsilon}{|1-2b|}. \tag{6}$$

Then (4), (5) and (6) yield

$$\int_0^{\pi/2} \frac{\sin^2(\theta)}{D(\theta)}\, d\theta \leq \frac{\frac{\pi}{2}}{\cos\left(\frac{\pi}{2} - \varepsilon\right)} + \frac{\varepsilon}{|1-2b|}. \tag{7}$$

Combining (1) and (7), we have

$$\left| \frac{L(b) - L\left(\frac{1}{2}\right)}{b - \frac{1}{2}} \right| \leq \frac{16 \left| b - \frac{1}{2} \right| \frac{\pi}{2}}{\cos\left(\frac{\pi}{2} - \varepsilon\right)} + 8\varepsilon < 9\varepsilon$$

for $|b - \frac{1}{2}|$ sufficiently small. Since ε was arbitrary, we conclude that $L'(\frac{1}{2})$ exists and is equal to 0.

Also solved by MICHEL BATAILLE, Rouen, France; CON AMORE PROBLEM GROUP, The Danish U. of Education, Copenhagen, Denmark; CHIP CURTIS, Missouri Southern State C.; DMITRY FLEISCHMAN, Santa Monica, CA; WEIHU HONG, Clayton C. & State U.; STEPHEN KACZKOWSKI, Orange County C.C.; MOHAMMAD RIAZI-KERMANI, Fort Hays State U.; LI ZHOU, Polk C. C.; and the proposer. Five incomplete solutions were received.

A matrix operation

759. *Proposed by Götz Trenkler, University of Dortmund, Germany*

If $\mathbf{A}$ and $\mathbf{B}$ are n by n matrices over an arbitrary field F, define $\mathbf{A} \circ \mathbf{B}$ to be the matrix $\mathbf{A} + \mathbf{B} - \mathbf{AB}$. Find necessary and sufficient conditions on $\mathbf{A}$ such that the equation $\mathbf{A} \circ \mathbf{B} = \mathbf{B} \circ \mathbf{A} = \mathbf{0}$ has a solution, $\mathbf{B}$.

Solution by Jim Delaney, California Polytechnic State University, San Luis Obispo, CA

This occurs if and only if $\mathbf{I} - \mathbf{A}$ is nonsingular. Note that $\mathbf{A} \circ \mathbf{B} = \mathbf{I} - (\mathbf{I} - \mathbf{A})(\mathbf{I} - \mathbf{B})$ and that $\mathbf{A} \circ \mathbf{B} = \mathbf{0}$ iff $(\mathbf{I} - \mathbf{A})(\mathbf{I} - \mathbf{B}) = \mathbf{I}$. Thus, if $\mathbf{A} \circ \mathbf{B} = \mathbf{0}$, then $\mathbf{I} - \mathbf{A}$ is nonsingular. Conversely, if $\mathbf{I} - \mathbf{A}$ is nonsingular and $\mathbf{C} = (\mathbf{I} - \mathbf{A})^{-1}$, then $(\mathbf{I} - \mathbf{A})\mathbf{C} = \mathbf{C}(\mathbf{I} - \mathbf{A}) = \mathbf{I}$. Let $\mathbf{B} = \mathbf{I} - \mathbf{C}$. Then $\mathbf{C} = \mathbf{I} - \mathbf{B}$, so that

$$(\mathbf{I} - \mathbf{A})(\mathbf{I} - \mathbf{B}) = (\mathbf{I} - \mathbf{B})(\mathbf{I} - \mathbf{A}) = \mathbf{I} \quad \text{or equivalently} \quad \mathbf{A} \circ \mathbf{B} = \mathbf{B} \circ \mathbf{A} = \mathbf{0}.$$

Also solved by REZA AKHLAGHI, Big Sandy Community and Technical C.; MICHAEL ANDREOLI, Miami-Dade C. (North); MICHEL BATAILLE, Rouen, France; ADAM COFFMAN, Indiana U.-Purdue U.-Fort Wayne; ELLIOTT COHEN, Fontenay-sous-Bois, France; CON AMORE PROBLEM GROUP, The Danish U. of Education, Copenhagen, Denmark; CHIP CURTIS, Missouri Southern State C.; DMITRY FLEISCHMAN, Santa Monica, CA; OVIDIU FURDUI (student), Western Michigan U.; TOMMY GOEBELER, Ursinus C.; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; EUGENE A. HERMAN, Grinnell C.; THOMAS MATTMAN, California State U.-Chico; NORTHWESTERN U. MATH PROBLEM SOLVING GROUP; MOHAMMAD RIAZI-KERMANI, Fort Hays State U.; WILLIAM SEAMAN, Albright C.; JOHN HENRY STEELMAN, Indiana U. of Pennsylvania; NORA THORNBER, Raritan Valley C.C.; XIAOSHEN WANG, U. of Arkansas-Little Rock; GREGORY P. WENE, U. of Texas-San Antonio; LI ZHOU, Polk C. C.; and the proposer.

An inequality for an abstract function

760. *Proposed by Arthur L. Holshouser, Charlotte, NC*

Suppose that f is a given function from the positive integers to the non-negative integers. We define a function g, whose domain is the non-negative integers, as follows: $g(0) = \infty$ and for n a positive integer, $g(n)$ is defined recursively by

$$g(n) \text{ is the smallest } x \text{ in } \{1, 2, \ldots, n\} \text{ such that } f(n) < g(n - x).$$

Note that $g(1) = 1$.

(a) If $f(n)$ is the largest power of 2 that divides n, find $g(n)$.

(b) Prove that for any f, if n is a positive integer and $1 \le x \le g(n) - 1$, then $g(n - x) \le g(n) - x$.

Solution by John Henry Steelman, Indiana University of Pennsylvania, Indiana, Pennsylvania

(a) We will show that $g(n) = f(n)$ for all $n > 0$. Clearly $g(1) = 1 = f(1)$, so we proceed by induction. Suppose that $g(m) = f(m)$ whenever $0 < m < n$. Note that n