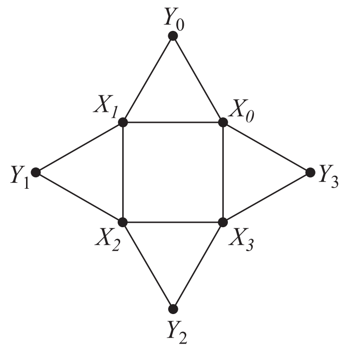
Matchings in a certain family of graphs **February 2020**

2090. *Proposed by Gregory Dresden, Washington & Lee University, Lexington, VA.*

Recall that a *matching* of a graph is a set of edges that do not share any vertices. For example, C_4, the cyclic graph on four vertices (i.e., a square), has seven matchings: the empty set, single edges (four of these), or pairs of opposite edges (two of these).
Let G_n be the (undirected) graph with vertices x_i and y_i, $0 \le i \le n-1$, and edges $\{x_i, x_{i+1}\}$, $\{x_i, y_i\}$, and $\{y_i, x_{i+1}\}$, $0 \le i \le n-1$, where the indices are to be taken modulo n. For example, G_4 is shown below. Determine the number of matchings of G_n.



Solution by the George Washington University Problems Group, George Washington University, Washington, DC.
The answer is 3^n. To see this, let $S = \{-1, 0, 1\}^n$, a set whose cardinality is clearly 3^n. We show that there is a bijection ϕ from S to the set of matchings of G_n. Let $a = (a_1, \ldots, a_n)$ be an element of S. We define $\phi(a)$ as follows:

$$\{x_i, x_{i+1}\} \in \phi(a) \text{ if and only if } a_i = 1 \text{ and } a_{i+1} = -1,$$
$$\{x_i, y_i\} \in \phi(a) \text{ if and only if } a_i = 1 \text{ and } a_{i+1} \ne -1, \text{ and}$$
$$\{x_{i+1}, y_i\} \in \phi(a) \text{ if and only if } a_i \ne 1 \text{ and } a_{i+1} = -1.$$

We now check that $\phi(a)$ is indeed a matching. The edges incident to y_i are not both in $\phi(a)$, since $\{x_i, y_i\} \in \phi(a)$ requires $a_i = 1$ but $\{x_{i+1}, y_i\} \in \phi(a)$ requires $a_i \ne 1$. Also, among the four edges incident to x_i, at most one can be chosen for $\phi(a)$, since including $\{x_i, x_{i-1}\}$, $\{x_i, y_{i-1}\}$, $\{x_i, y_i\}$, and $\{x_i, x_{i+1}\}$ require, respectively, the four mutually exclusive conditions (1) $a_i = -1$ and $a_{i-1} = 1$, (2) $a_i = -1$ and $a_{i-1} \ne 1$, (3) $a_i = 1$ and $a_{i+1} \ne -1$, and (4) $a_i = 1$ and $a_{i+1} = -1$.

Given a matching M, there is a unique $a \in S$ so that M is $\phi(a)$. To see this, let $a_i = 1$ if M contains $\{x_i, x_{i+1}\}$ or $\{x_i, y_i\}$, let $a_i = -1$ if M contains $\{x_{i-1}, x_i\}$ or $\{x_i, y_{i-1}\}$, and let $a_i = 0$ if x_i is not the endpoint of any edge in M. This element $a \in S$ is the only element in $\phi^{-1}(M)$. Hence ϕ is bijective.

Also solved by Elton Bojaxhiu (Germany) and Enkel Hysnelaj (Australia), Robert Calcaterra, Jiakang Chen, Eddie Cheng; Serge Kruk; Li Li & László Lipták (jointly), José H. Nieto (Venezuela), Kishore Rajesh, Edward Schmeichel, John H. Smith, and the proposer. There was one incomplete or incorrect solution.

Answers

Solutions to the Quickies from page 72.