$$n\sum_{i=0}^{n+1} a_i^4 = 2\sum_{0\le i<j\le n+1} a_i^2 a_j^2,$$

which is the desired relation.

Also solved by Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Albert Stadler (Switzerland), and the proposer. There were two incomplete or incorrect solutions.

Find the normalizer **April 2021**

2120. *Proposed by Gregory Dresden, Jackson Gazin (student), and Kathleen McNeill (student), Washington & Lee University, Lexington, VA.*

Recall that the normalizer of a subgroup H of G is defined as

$$N_G(H) = \left\{g \in G | ghg^{-1} \in H \text{ for all } h \in H\right\}.$$

Determine $N_G(H)$, when $G = GL_2(\mathbb{R})$, the group of all invertible 2×2 matrices with real entries, and

$$H = SO_2(\mathbb{R}) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle| \theta \in \mathbb{R} \right\}.$$

Solution by Eugene A. Herman, Grinnell College, Grinnell, IA.

More generally, for any $n \ge 1$, let $G = GL_n(\mathbb{R})$ and $H = SO_n(\mathbb{R})$, the subgroup of $O_n(\mathbb{R})$, the group of orthogonal matrices, consisting of matrices whose determinant is 1. We will show that

$$N_G(H) = \{aU \,|a \in \mathbb{R} - \{0\}, U \in O_n(\mathbb{R})\}.$$

Suppose $A = aU$, where $a \neq 0$ and U is orthogonal. Then for any $M \in SO_n(\mathbb{R})$,

$$AMA^{-1} = aUM\frac{1}{a}U^{-1} = UMU^{-1}.$$

Since

$$\det(UMU^{-1}) = \det(U)\det(M)/\det(U) = 1,$$

and the product of orthogonal matrices is orthogonal, we see that $AMA^{-1} \in SO_n(\mathbb{R})$.

For the converse, we use a polar decomposition. For $A \in N_G(H)$, write $A = PU$, where P is positive-definite and U is orthogonal. For any $M \in SO_n(\mathbb{R})$, let $N = U^{-1}MU$. Then $N \in SO_n(\mathbb{R})$, so $ANA^{-1} \in SO_n(\mathbb{R})$. But

$$ANA^{-1} = P(UNU^{-1})P^{-1} = PMP^{-1},$$

so $P \in N_G(H)$. Therefore, it remains only to determine which positive-definite matrices are in the normalizer. Now every positive-definite matrix can be written as $P = VDV^{-1}$, where $D = \operatorname{diag}(d_1, \ldots, d_n)$ is a diagonal matrix with $d_i > 0$ and $V \in O_n(\mathbb{R})$. For any $M \in SO_n(\mathbb{R})$, let $N = VMV^{-1}$. Then $B = PNP^{-1} \in SO_n(\mathbb{R})$ and

$$B = VDMD^{-1}V^{-1} \in SO_n(\mathbb{R}), \text{ so } DMD^{-1} = V^{-1}BV \in SO_n(\mathbb{R}).$$

Therefore, $D \in N_G(H)$.

For $k > 1$, let $M_k = [m_{ij}]$, where

$$m_{11} = 0, m_{1k} = -1, m_{k1} = 1, m_{kk} = 0, m_{ii} = 1\ (i \neq 1, k), \text{ and } m_{i,j} = 0 \text{ otherwise.}$$

Then $R \in SO_n(\mathbb{R})$ and the first column of DRD^{-1} consists of zeros except the kth entry, which is d_k/d_1. Since DRD^{-1} is orthogonal, this column must have length 1, which means that $d_k = d_1$ for all $k > 1$. Therefore D is a positive multiple of the identity, and so A is a multiple of an orthogonal matrix.

Note: The same proof works for the complex version. In that case, $G = GL_n(\mathbb{C})$ and $H = SU_n(\mathbb{C})$, where the latter is the group of $n \times n$ unitary matrices whose determinant equals 1. Then $N_G(H)$ is the group of all nonzero complex multiples of $n \times n$ unitary matrices.

Also solved by Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Robert Calcaterra, Eagle Problem Solvers (Georgia Southern University), John Fitch, Dmitry Fleischman, Mark Kaplan & Michael Goldenberg, Koopa Tak Lun Koo (Hong Kong), Didier Pinchon (France), Albert Stadler (Switzerland) and the proposers. There were two incomplete or incorrect solutions.

Answers

Solutions to the Quickies from page 158.

A1119. The aces divide the 48 other cards into 5 "urns", with a, b, c, d, and e non-aces in them, respectively. The position of the third ace is equal to $a + b + c + 3$, so the expected value of its position is $E[a + b + c + 3]$. By linearity of expectation, this is $E[a] + E[b] + E[c] + 3$. Because a non-ace is equally likely to be placed in any of the five "urns", $E[a] = \ldots = E[e]$. Since $E[a + b + c + d + e] = 48$, we have $E[a] = \ldots = E[e] = \frac{48}{5}$.

Therefore the expected value is

$$3 \cdot \frac{48}{5} + 3 = \frac{159}{5}.$$

A1120. Let S, S_1, S_2, and S_3 be the areas of $\triangle ABC$, $\triangle XBC$, $\triangle XCA$, and $\triangle XAB$, respectively. Let h_2 and h_3 be the heights of $\triangle XCA$ and $\triangle XAB$ with $\overline{AX}$ as base. Let θ be the angle between $\overleftrightarrow{AX}$ and $\overleftrightarrow{BC}$. Then

$$S_2 + S_3 = \frac{1}{2}(h_2 + h_3)\, R_1 = \frac{1}{2} a \sin\theta\, R_1 \leq \frac{1}{2} a R_1.$$

Similar arguments give

$$S_3 + S_1 \leq \frac{1}{2} b R_2 \text{ and } S_1 + S_2 \leq \frac{1}{2} c R_3.$$

Therefore

$$\frac{1}{2} a R_1 + \frac{1}{2} b R_2 + \frac{1}{2} c R_3 \geq (S_2 + S_3) + (S_3 + S_1) + (S_1 + S_2) = 2S = r(a + b + c)$$

and the result follows.

Equality holds if and only if the line through a vertex and X and the line containing the side opposite the vertex are perpendicular. In other words, X must be the orthocenter of the triangle, which must be acute in order for X to lie in its interior.

Note. Let O and R be the circumcenter and the circumradius for a given acute triangle. Since $R_1 = R_2 = R_3 = R$, we obtain Euler's inequality $R \geq 2r$.