

Find the area of the checkerboard pattern**December 2022****2160.** *Proposed by Gregory Dresden, Washington & Lee University, Lexington, VA.*

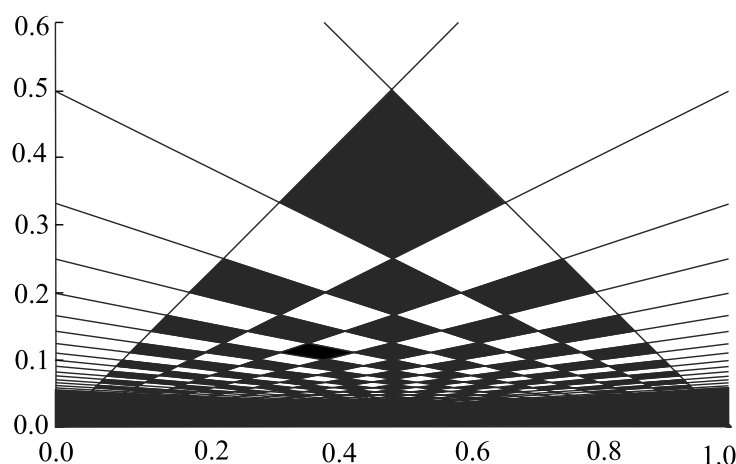
Consider the lines

$$y = x/1, y = x/2, y = x/3, y = x/4, \dots$$

and the lines

$$y = (1 - x)/1, y = (1 - x)/2, y = (1 - x)/3, y = (1 - x)/4, \dots,$$

which intersect to form an infinite number of quadrilaterals. Starting with the lozenge at the top, shade every other quadrilateral, as shown in the figure.



Find the total area of all the shaded quadrilaterals.

*Solution by Clayton Coe (student), Cal Poly Pomona, Pomona, CA.*The total area is $2 \ln 2 - \frac{5}{4}$.

Let $y = x/n$ be the equation of line L_n , and $y = (1 - x)/k$ be the equation of line M_k , where $n, k \geq 1$. Observe that

$$L_n \cap M_k = \left(\frac{n}{n+k}, \frac{1}{n+k} \right).$$

The set of vertices of any of these tiles is

$$\{L_n \cap M_k, L_n \cap M_{k+1}, L_{n+1} \cap M_{k+1}, L_{n+1} \cap M_k\}.$$

Note that $L_{n+1} \cap M_k$ and $L_n \cap M_{k+1}$ have the same y -coordinate. Therefore, we can calculate the area of a quadrilateral to be the sum of the area of two triangles with horizontal bases. Doing so, we find the area to be $hw/2$, where h is the difference between the y -coordinates of $L_n \cap M_k$ and $L_{n+1} \cap M_{k+1}$, and w is the difference between the x -coordinates of $L_n \cap M_{k+1}$ and $L_{n+1} \cap M_k$.

Let $A(n, k)$ denote the area of a single tile, with uppermost vertex $L_n \cap M_k$. We therefore have

$$A(n, k) = \frac{1}{2} \left(\frac{1}{n+k} - \frac{1}{n+k+2} \right) \left(\frac{n+1}{n+k+1} - \frac{n}{n+k+1} \right)$$

$$\begin{aligned}
&= \frac{1}{2} \left(\frac{2}{(n+k)(n+k+2)} \right) \left(\frac{1}{n+k+1} \right) \\
&= \frac{1}{(n+k)(n+k+1)(n+k+2)}.
\end{aligned}$$

Note that each black quadrilateral has uppermost vertex $L_n \cap M_k$ with $n+k$ even. Letting $n+k=2m$,

$$A(n, k) = \frac{1}{2m(2m+1)(2m+2)}.$$

In each horizontal row of quadrilaterals, $n+k$ is constant, and there are $n+k-1=2m-1$ quadrilaterals in that row. Consequently, the sum of the areas of all black quadrilaterals is

$$S = \sum_{m=1}^{\infty} \frac{2m-1}{2m(2m+1)(2m+2)}$$

Observe that the above sum is absolutely convergent because it is comparable to a p -series, with $p=2$. We perform a partial fraction decomposition, yielding

$$S = \sum_{m=1}^{\infty} \left(\frac{-1/2}{2m} + \frac{2}{2m+1} + \frac{-3/2}{2m+2} \right)$$

Because of absolute convergence, we may shift the index of the first term of the summand, and obtain

$$\begin{aligned}
S &= -\frac{1}{4} + \sum_{m=1}^{\infty} \left(\frac{-1/2}{2(m+1)} + \frac{2}{2m+1} + \frac{-3/2}{2m+2} \right) \\
&= -\frac{1}{4} + \sum_{m=1}^{\infty} \left(\frac{2}{2m+1} - \frac{2}{2m+2} \right) \\
&= -\frac{1}{4} + 2 \sum_{m=0}^{\infty} \left(\frac{1}{2m+1} - \frac{1}{2m+2} \right) - 2 \left(1 - \frac{1}{2} \right) \\
&= -\frac{5}{4} + 2 \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} \\
&= -\frac{5}{4} + 2 \ln 2
\end{aligned}$$

Also solved by Farrukh Rakhimjanovich Ataev (Uzbekistan), Chip Curtis, Eagle Problem Solvers, Dmitry Fleischman, Eugene A. Herman, Walther Janous (Austria), Do Hyun Lee (South Korea), Chrysostom G. Petalas (Greece), William Reil, Volkhard Schindler (Germany), Edward Schmeichel, Paul K. Stockmeyer, Maria van der Walt, and the proposer.