Areas Within Cyclic-Harmonic Curves

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January 25, 2025

Shown here in Figure 1 are two copies of the graph of $r = 2 + \cos 11\theta/6$, which is known as a cyclic-harmonic curve.

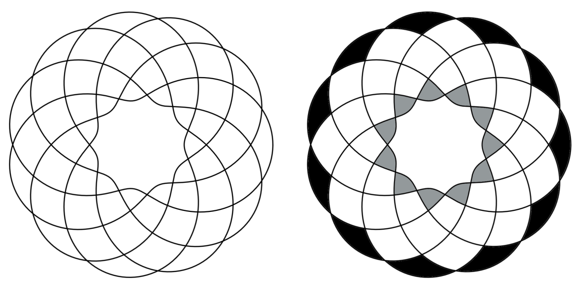

Figure 1: Two copies of a typical cyclic-harmonic curve.

These pictures have 11-fold rotational symmetry and five concentric rings, two of which we have highlighted on the right in black and gray. We will show that the total black area in Figure 1 is $24 - 45\sqrt{3}/4$ and the total gray area is $24 - 51\sqrt{3}/4$.

If we now consider the generic $r = a + \cos p\theta/q$ with $a > 1$ and with $p, q \geq 2$ relatively prime integers, then the graph will change quite a bit; our 11-fold rotational symmetry will become p-fold symmetry, and instead of five rings our generic curve will have $q - 1$ rings. None the less, we can still find expressions for the two highlighted areas (black for the outer ring, and gray for the inner one). While these new expressions will not be as simple as in our earlier example, we discover to our surprise that these areas are independent of p. Next, we show (also to our surprise) that the difference of these total areas is independent of p and a, and achieves a maximum value of 4 when $q = 4$. Finally, we calculate the ratio of these two areas and we show that as $q \to \infty$ the ratio goes to $(a + 1)/(a - 1)$.

1. Historical Background

As mentioned above, these curves $r = a + \cos p\theta/q$ are called *cyclic-harmonic curves*, and they were studied rather extensively by Robert E. Moritz in the early 1900's. Moritz created a simple mechanism [1, Figure 6] to draw these and other curves for different values of p and q, and he also wrote several articles [2, 3] discussing various properties of these curves. His work was continued by the Jesuit astronomer and physics professor William Rigge who spent ten years [4] building a more intricate mechanical device which is shown in Figure 2.

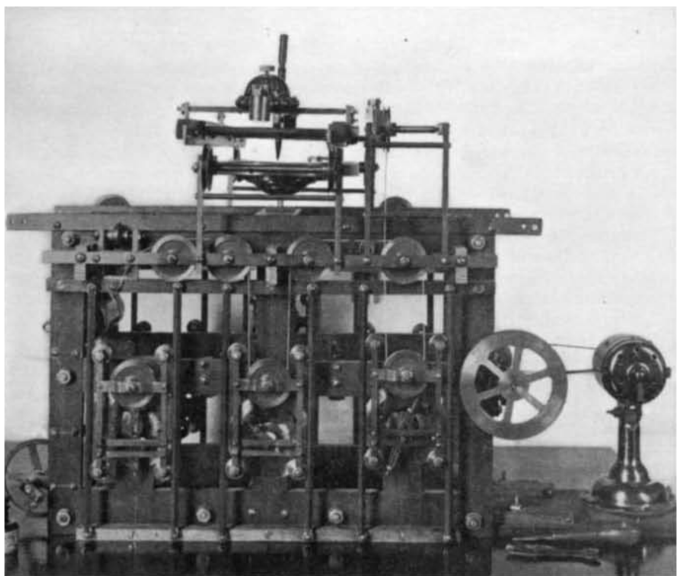

Figure 2: Rigge's mechanical device.

This photograph is taken from a supplement to *Scientific American* [5] from the year 1918. The machine is about two feet wide and just over a foot tall, and would have been a marvel in its time. Rigge used it to produce intricate polar and rectangular curves [6, 7, 8] and even stereoscopic curves [9] which still dazzle the eye today. Some of his (two-dimensional) curves are shown here in Figure 3.

We can see distant echoes of Rigge's device in the popular Spirograph toy which can draw some simple roulettes. We note with pleasure that Rigge's mechanical device has survived to the modern day. It is located in the aptly-named Rigge Science Building, home to the physics department at Creighton University where Rigge spent many decades of his life.

For whatever reason, neither Moritz nor Rigge studied the *areas* of these cyclic-harmonic curves, and so we are delighted to resurrect these long-dormant equations and to share our new discoveries about these old curves.

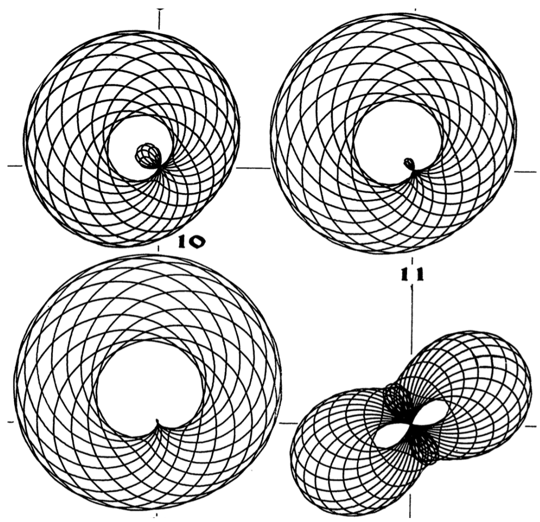

Figure 3: Mechanically-drawn curves from 1920.

2. Our Main Result

Theorem 1. *For $r = a + \cos p\theta/q$ with $a > 1$ and with $p, q \geq 2$ relatively prime, the total area of the outer ring of the graph is*

$$A_{\text{black}} = \frac{q}{4}\left(16a\sin\frac{\pi}{q} - (8a-2)\sin\frac{2\pi}{q} - \sin\frac{4\pi}{q}\right) \tag{1}$$

and the total area of the inner ring is

$$A_{\text{gray}} = \frac{q}{4}\left(16a\sin\frac{\pi}{q} - (8a+2)\sin\frac{2\pi}{q} + \sin\frac{4\pi}{q}\right). \tag{2}$$

We note that when $q = 6$, then equations (1) and (2) give us

$$A_{\text{black}} = 24 - 45\sqrt{3}/4$$
$$A_{\text{gray}} = 24 - 51\sqrt{3}/4,$$

the areas of the two regions from Figure 1 as mentioned in the introduction to this article.

Before venturing into the proof of Theorem 1, let us first discuss some of the conclusions we can draw from it. First, we note that the *difference* of equations (1) and (2) is

$$A_{\text{black}} - A_{\text{gray}} = \frac{q}{4}\left(4\sin\frac{2\pi}{q} - 2\sin\frac{4\pi}{q}\right).$$

This is indeed independent of both p and a, and it simplifies nicely to

$$A_{\text{black}} - A_{\text{gray}} = q \sin \frac{2\pi}{q} \left(1 - \cos \frac{2\pi}{q}\right). \tag{3}$$

It is a straightforward optimization problem to show that the expression in (3) achieves a maximum for integer $q \geq 2$ at $q = 4$, with the maximum such value being 4. To illustrate this surprising result, Figure 4 shows three graphs of $r = a + \cos p\theta/4$ with differing values for a and p. In every case, the total black area is exactly four more than the total gray area.

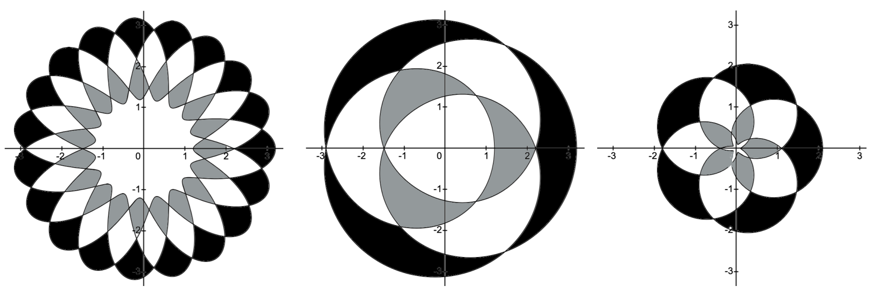

Figure 4: For $q = 4$, the area of the outer ring is always four more than the area of the inner ring.

As for the *ratio* of equations (1) and (2), we see that this becomes

$$\frac{A_{\text{black}}}{A_{\text{gray}}} = \frac{16a \sin \dfrac{\pi}{q} - (8a - 2) \sin \dfrac{2\pi}{q} - \sin \dfrac{4\pi}{q}}{16a \sin \dfrac{\pi}{q} - (8a + 2) \sin \dfrac{2\pi}{q} + \sin \dfrac{4\pi}{q}}.$$

To find the limit of this ratio as $q \to \infty$, we can make it easier to calculate if we replace q with $1/x$ and we replace $q \to \infty$ with $x \to 0$. This gives us

$$\lim_{q \to \infty} \frac{A_{\text{black}}}{A_{\text{gray}}} = \lim_{x \to 0} \frac{16a \sin \pi x - (8a - 2) \sin 2\pi x - \sin 4\pi x}{16a \sin \pi x - (8a + 2) \sin 2\pi x + \sin 4\pi x},$$

and while it is tempting to apply L'Hôpital's Rule, it is actually easier to simply expand each term using Taylor series around $x = 0$. This gives us

$$\lim_{q \to \infty} \frac{A_{\text{black}}}{A_{\text{gray}}} = \lim_{x \to 0} \frac{8\pi^3(a + 1)x^3 - 2\pi^5(a + 4)x^5 + O(x^7)}{8\pi^3(a - 1)x^3 - 2\pi^5(a - 4)x^5 + O(x^7)}$$

and after dividing top and bottom by $8\pi^3 x^3$ this limit gives us $(a + 1)/(a - 1)$, as desired. See Figure 5.

We summarize the above discussions in the following corollary.

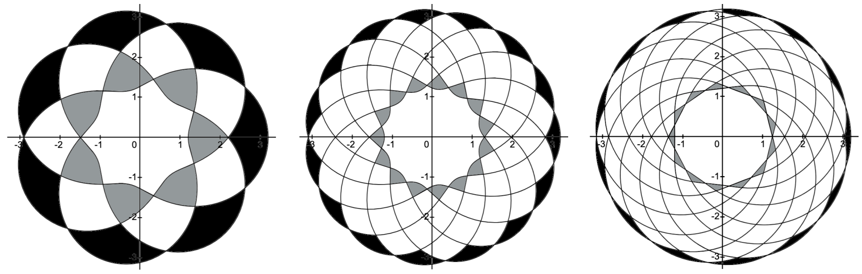

Figure 5: As $q \to \infty$, the area of the outer ring approaches $(a+1)/(a-1)$ times the area of the inner ring.

Corollary 1. *For $r = a + \cos p\theta/q$ as seen in Theorem 1, the difference between the total area of the outer ring and the total area of the inner ring depends only on q and is equal to*

$$A_{\text{black}} - A_{\text{gray}} = q \sin \frac{2\pi}{q} \left(1 - \cos \frac{2\pi}{q}\right). \tag{4}$$

This difference is maximal (and equal to 4) when $q = 4$. As for the ratio, it is dependent only on a and q, and as we take q going to infinity we have

$$\lim_{q\to\infty} \frac{A_{\text{black}}}{A_{\text{gray}}} = \frac{a+1}{a-1}. \tag{5}$$

We are now ready for the proof of our main result.

Proof of Theorem 1. We note that the all of our figures seem to have both *reflection symmetry* (across the x-axis) and *rotation symmetry* (in Figure 1, rotating by $2\pi/11$). To prove this for the general case with $r(\theta) = 2 + \cos p\theta/q$, we note that $r(\theta) = r(-\theta)$ giving us the reflection symmetry. As for the rotation symmetry, we will show that rotating the graph by $2\pi/p$ will give us the same picture (and thus will give us p-fold rotational symmetry). To do this, we begin by noting that direct substitution gives us

$$r(\theta) \;=\; r\left(\theta + \frac{1q \cdot 2\pi}{p}\right) \;=\; r\left(\theta + \frac{2q \cdot 2\pi}{p}\right) \;=\cdots=\; r\left(\theta + \frac{(p-1)q \cdot 2\pi}{p}\right).$$

Since p and q are relatively prime, the set $\{1q, 2q, \ldots, (p-1)q\}$ is a complete nonzero residue set mod p, and so one of them (let's call it jq) is equivalent to 1 mod p, which means $jq = kp + 1$ for some k. This means that

$$r(\theta) = r\left(\theta + \frac{jq \cdot 2\pi}{p}\right) = r\left(\theta + \frac{(kp+1) \cdot 2\pi}{p}\right) = r\left(\theta + k2\pi + \frac{2\pi}{p}\right),$$

and since the angle $\theta + k2\pi + 2\pi/p$ is at the same position on the polar plane as the angle $\theta + 2\pi/p$, then when we rotate the graph by the angle $2\pi/p$ we get the same graph, thus giving us our desired p-fold rotational symmetry. (This is a simplified version of the symmetry argument given in [3, Theorem 2]).

We note that our rotational symmetry is not *more* than p-fold rotational symmetry, because our polar graph $r(\theta) = 2 + \cos p\theta/q$ achieves its maximum distance from the origin only when $\theta = 0$ and $\theta = 1q \cdot 2\pi/p$ and $\theta = 2q \cdot 2\pi/p$ and so on, up to $\theta = (p-1)q \cdot 2\pi/p$ after which we start to repeat the angles. Hence, we have at most, and thus exactly, p-fold rotational symmetry.

Next, we need to understand the self-intersections for this polar graph. This, too, can be found in [3] but we provide here a simpler and self-contained argument. Because p and q are relatively prime, our polar graph will need to wrap around the origin q times before returning to the original starting configuration. Another way of thinking about this is to recognize that the single graph $r = 2 + \cos(p\theta/q)$ for θ in $[0, q2\pi]$ is the same as the q separate graphs

$$r_0 = 2 + \cos\left(\frac{p\theta}{q}\right), \quad r_1 = 2 + \cos\left(\frac{p(\theta + 1 \cdot 2\pi)}{q}\right),$$

all the way up to

$$r_{q-1} = 2 + \cos\left(\frac{p(\theta + (q-1) \cdot 2\pi)}{q}\right),$$

with θ in $[0, 2\pi]$ for all q of these graphs. See Figure 7.

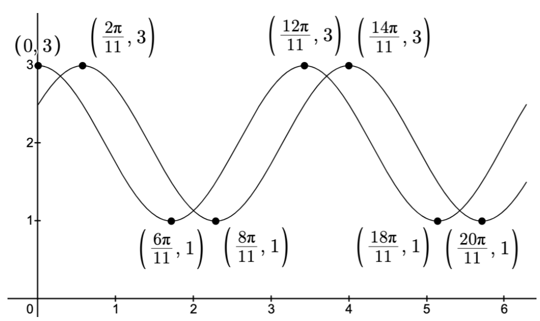

Figure 6: Cartesian graphs of $r_0 = 2 + \cos(11\theta/6)$ and $r_1 = 2 + \cos(11(\theta + 2\pi)/6)$.

If we plot these graphs on a *Cartesian* plane, we can immediately recognize that they are distinct and we can see where they intersect. Figure 6 shows the graphs for just r_0 and r_1 with $p/q = 11/6$, and since r_0 achieves its extremums at $\theta = 6n\pi/11$ and likewise r_1 at $\theta = (6n+2)\pi/11$, then the intersections for r_0 and r_1 will occur midway between them.

Figure 7 shows all six graphs $r_0, r_1, \ldots, r_5$. We see that the intersections for all these graphs can only occur at θ some integer multiple of $\pi/11$. Moving to the general case, we conclude that the self-intersections for $r = 2 + \cos(p\theta/q)$ will occur when θ is one of the $2p$ integer multiples of π/p.

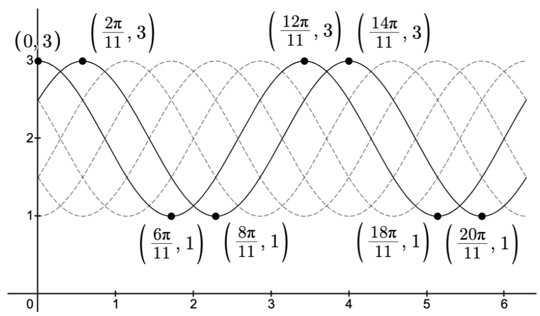

Figure 7: Intersections occur when θ is an integer multiple of $\pi/11$.

We are now ready to calculate the area of the outside (black) regions. If we focus our attention on the sector between $\theta = 0$ and $\theta = 2\pi/p$, we have self-intersections at the three angles $\theta = 0$, $\theta = \pi/p$, and $\theta = 2\pi/p$, as shown by the three rays in Figure 8.

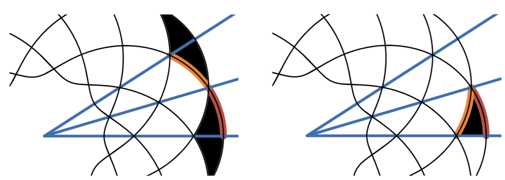

Figure 8: Finding the area of the outer (black) ring.

The total outside (black) area is made up of p copies of these black kite shapes (two of them are shown in Figure 8 on the left). The outer curve, shown in red, is given by $r = 2 + \cos(p\theta/q)$ for θ in $[0, \pi/p]$, and if we extend θ to the interval $[\pi/p, 2\pi/p]$ then we get the orange part of the curve in the picture on the left of Figure 8. Thanks to the rotation and reflexion symmetry, this has the same shape as the orange curve on the right. All this is to say that the area of that half-kite black shape on the right of Figure 8 is

$$\int_0^{\pi/p} \frac{1}{2} \left(a + \cos(p\theta/q)\right)^2 \, d\theta - \int_{\pi/p}^{2\pi/p} \frac{1}{2} \left(a + \cos(p\theta/q)\right)^2 \, d\theta.$$

We can easily calculate this, and then multiply it by $2p$, to obtain the following expression for the total black area, which matches Equation (1) in the statement of Theorem 1:

$$A_{\text{black}} = \frac{q}{4} \left(16a \sin\frac{\pi}{q} - (8a - 2) \sin\frac{2\pi}{q} - \sin\frac{4\pi}{q} \right).$$

Surprisingly, this formula does not depend on p.

A similar argument applies for calculating the area of the inside (gray) regions. This time we look at the sector between $\theta = q\pi/p$ and $\theta = (q+2)\pi/p$, because our graph is at its minimum when $\theta = q\pi/p$. In this sector, we have self-intersection at the three angles $\theta = q\pi/p$, $\theta = (q+1)\pi/p$, and $\theta = (q+2)\pi/p$, as shown by the three rays in Figure 9.

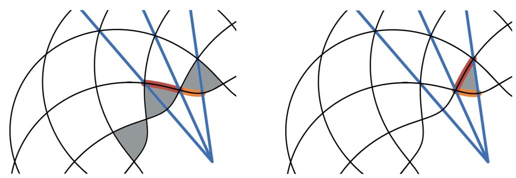

Figure 9: Finding the area of the inner (gray) ring.

The total inside (gray) area is made up of p copies of these gray kite shapes (three of them are shown in Figure 9 on the left). The inner curve, shown in orange, is given by $r = 2 + \cos(p\theta/q)$ for θ in $[q\pi/p, (q+1)\pi/p]$, and if we extend θ to the interval $[(q+1)\pi/p, (q+2)\pi/p]$ then we get the red part of the curve in the picture on the left of Figure 9. Thanks to the rotation and reflexion symmetry, this has the same shape as the red curve on the right. All this is to say that the area of that half-kite gray shape on the right of Figure 9 is

$$\int_{(q+1)\pi/p}^{(q+2)\pi/p} \frac{1}{2} \left(a + \cos(p\theta/q)\right)^2 \, d\theta - \int_{q\pi/p}^{(q+1)\pi/p} \frac{1}{2} \left(a + \cos(p\theta/q)\right)^2 \, d\theta.$$

We can easily calculate this, and multiply by $2p$, to obtain the following expression for the total gray area, which nicely matches Equation (2):

$$A_{\text{gray}} = \frac{q}{4} \left(16a \sin\frac{\pi}{q} - (8a+2)\sin\frac{2\pi}{q} + \sin\frac{4\pi}{q} \right).$$

Once again, this formula does not depend on p.

Having now established Equations (1) and (2) of Theorem 1, this concludes our proof. □

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Abstract

The cyclic-harmonic curves $r = 2 + \cos p\theta/q$ are related to the familiar roses $r = \cos n\theta$ we all remember from calculus as our first introduction to polar coordinate graphs. These cyclic-harmonic curves were studied rather extensively by Robert Moritz and William Rigge in the 1910's and 20's and have been mostly dormant since then. We bring them back to life and we share some new discoveries related to their areas.