Self-Convolutions of Generalized Narayana Numbers

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ABSTRACT
For the Fibonacci numbers F_n, we have the self-convolution formula $5\sum_{i=0}^{n} F_i F_{n-i} = (2n)F_{n+1} - (n+1)F_n$. We find the corresponding self-convolution formula for the Narayana numbers R_n which satisfy $R_n = R_{n-1} + R_{n-3}$, and then generalize it to the k-step Narayana numbers $\mathcal{R}_n$ with order-k recurrence formula $\mathcal{R}_n = \mathcal{R}_{n-1} + \mathcal{R}_{n-k}$.

KEYWORDS
Fibonacci, Narayana, convolution

1. Introduction

We begin with the Fibonacci numbers, defined as

$$F_0 = 0,\ F_1 = 1,\ \text{and } F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2.$$

For the following collection of formulas, the sum on the left is called the *self-convolution* of the Fibonacci numbers.

$$\begin{aligned} 5\sum_{i=0}^{n} F_i F_{n-i} &= (2n)F_{n+1} - (n+1)F_n \qquad (1.1) \\ &= (n-1)F_n + (2n)F_{n-1} \\ &= (n-1)F_{n+1} + (n+1)F_{n-1} \\ &= nL_n - F_n \end{aligned}$$

Most of these can be found at A001629 on the On-Line Encyclopedia of Integer Sequences (OEIS) [8] and can also be found in various forms at [2, Identity 58], [7, Theorem 1], [10, p. 183], [12, Corollary 1], and in many other papers. We note that the L_n in the last formula represents the nth Lucas number.

For the self-convolution of the Lucas numbers L_n, we have

$$\sum_{i=0}^{n} L_i L_{n-i} = (n+1)L_n + 2F_{n+1} \qquad (1.2)$$

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which appears in [2, Identity 57] and [9, Corollary 10]. The Fibonacci numbers and the Lucas numbers are two specific (and famous) examples of second-order recurrences. Adegoke, Akerele, and Frontczak in [1] use variations on Binet's formula to give self-convolution formulas for general second-order recurrence sequences; their paper includes equations (1.1) and (1.2), above, along with many others.

Aside from [4, Theorem 6] which generalizes equation (1.2) to order-k Lucas-type sequences and [6, equation (52)] which covers third-order recurrences, there are not many results on the self-convolutions of higher-order recurrence sequences. This paper aims to partially fill that gap. In particular, we will consider the order-k recurrence sequences that are natural generalizations of the Narayana sequence (which is itself a generalization of the Fibonacci numbers).

2. The Narayana Sequence

We now define the Narayana numbers R_n as

$$R_0 = 0,\ R_i = 1 \text{ for } i = 1, 2, \text{ and } R_n = R_{n-1} + R_{n-3} \text{ for } n \geq 3. \tag{2.1}$$

We can verify by hand that the following self-convolution equation seems to be true:

$$31 \sum_{i=0}^{n} R_i R_{n-i} = 9(n+1)R_{n+2} - 3(n+3)R_{n+1} - 2(n+2)R_n. \tag{2.2}$$

Equation (2.2) is indeed true, and can be derived from equation (52) in Rabonowitz's 1996 article [6], with a little bit of effort. Rabinowitz's equation covers a general third-order recurrence defined as

$$X_0 = X_1 = 0, X_2 = 1, \quad \text{and} \quad X_n = pX_{n-1} + qX_{n-2} + rX_{n-3} \quad \text{for } n \geq 3.$$

If we take $p = 1$, $q = 0$, and $r = 1$ to match equation (2.1), we eventually arrive at

$$\sum_{i=0}^{n} X_i X_{n-i} = \Big(6(n-2)X_{n+1} - 2nX_{n-1} + 3(n+1)X_{n-2}\Big)/31. \tag{2.3}$$

Since $X_n = R_{n-1}$ thanks to the slightly shifted initial values, then with a bit more manipulation we can transform equation (2.3) into equation (2.2).

3. The 4-step Narayana numbers

We can think of the Narayana numbers as being a "3-step" sequence because the last term in the recurrence equation (2.1) is R_{n-3}. With this in mind, we also define the "4-step" Narayana numbers S_n as

$$S_0 = 0,\ S_i = 1 \text{ for } 1 \leq i \leq 3, \text{ and } S_n = S_{n-1} + S_{n-4} \text{ for } n \geq 4. \tag{3.1}$$

We can verify (with some difficulty) that

$$283 \sum_{i=0}^{n} S_i S_{n-i} = 64(n+2)S_{n+3} - 16(n+5)S_{n+2} - 12(n+4)S_{n+1} - 9(n+3)S_n. \tag{3.2}$$

The constants 5, 31, and 283 that appear on the left of each of the self-convolution formulas (1.1), (2.2), and (3.2) might seem to be random, but we note that $\sqrt{5}$ is part of the golden ratio [2] related to the Fibonacci numbers, and likewise $\sqrt{31}$ for the "supergolden" ratio [3] associated with the Narayana numbers. Another insight comes from noting that 5, -31, and -283 are the discriminants of $1 - x - x^k$ for $k = 2$, 3, and 4. For our purposes, we make the observation that all three can be expressed as the sum of two powers:

$$5 = 2^2 + 1^1, \qquad 31 = 3^3 + 2^2, \qquad \text{and} \qquad 283 = 4^4 + 3^3. \tag{3.3}$$

Likewise, there seems to be a pattern to the coefficients on the right of each of those self-convolution formula. We can make this more clear if we re-write equation (3.2) as follows:

$$(4^4 + 3^3) \sum_{i=0}^{n} S_i S_{n-i} = 4^3(n+2)S_{n+3} - \Big(3^2(n+3)S_n + 3 \cdot 4(n+4)S_{n+1} + 4^2(n+5)S_{n+2}\Big). \tag{3.4}$$

This inspires us to re-write equation (2.2) in the same style as (3.4):

$$(3^3 + 2^2) \sum_{i=0}^{n} R_i R_{n-i} = 3^2(n+1)R_{n+2} - \Big(2^1(n+2)R_n + 3^1(n+3)R_{n+1}\Big). \tag{3.5}$$

We do the same with our Fibonacci formula in equation (1.1), giving us

$$(2^2 + 1^1) \sum_{i=0}^{n} F_i F_{n-i} = 2^1(n+0)F_{n+1} - \Big(1^0(n+1)F_n\Big). \tag{3.6}$$

We can now see that there is a common theme for all three self-convolution formulas. As we show in the next section, we can generalize this beyond the 3-step and 4-step Narayana numbers R_n and S_n, respectively.

4. Main Result

We define the k-step Narayana numbers $\mathcal{R}_n$ as

$$\mathcal{R}_0 = 0,\ \mathcal{R}_i = 1 \text{ for } 1 \le i \le k-1, \text{ and } \mathcal{R}_n = \mathcal{R}_{n-1} + \mathcal{R}_{n-k} \text{ for } n \ge k. \tag{4.1}$$

When $k = 2$ we recapture the Fibonacci numbers F_n, and for $k = 3$ and $k = 4$ we obtain R_n and S_n, respectively.

Our main result is as follows.

Theorem 4.1. *For $k \geq 2$ fixed, and with $\mathcal{R}_n$ representing the k-step Narayana numbers defined above in (4.1), we have*

$$\begin{aligned}\left(k^k + (k-1)^{k-1}\right) \sum_{i=0}^{n} \mathcal{R}_i \mathcal{R}_{n-i} \\ = k^{k-1}(n+k-2)\mathcal{R}_{n+k-1} - \sum_{j=0}^{k-2} \left(k^j \cdot (k-1)^{k-2-j}\right)(n+k+j-1)\mathcal{R}_{n+j}. \quad (4.2)\end{aligned}$$

It is important to note that this theorem applies for any and all $k \geq 2$. For example, if we want a formula for the self-convolution of the 6-step Narayana numbers, which we define as

$$U_0 = 0,\ U_i = 1 \text{ for } 1 \leq i \leq 5, \text{ and } U_n = U_{n-1} + U_{n-6} \text{ for } n \geq 6, \quad (4.3)$$

then Theorem 4.1 tells us that

$$(6^6 + 5^5) \sum_{i=0}^{n} U_i U_{n-i} = 6^5(n+4)U_{n+5} - \sum_{j=0}^{4} 6^i 5^{4-i}(n+5+j)U_{n+j}.$$

5. Technical Lemmas

Here are some lemmas that we will need for the proof of our Theorem 4.1.

Lemma 5.1. *For $\theta \neq 0$ or 1, we have*

$$\sum_{i=0}^{m} \theta^i = \frac{\theta^{m+1} - 1}{\theta - 1} \qquad \textit{and} \qquad \sum_{i=0}^{m} i\theta^i = \frac{\theta(1-\theta^m)}{(\theta-1)^2} + \frac{m\theta^{m+1}}{\theta - 1}. \quad (5.1)$$

Proof. The first sum is a finite geometric series, and the second sum follows from equation 1.2.2.3 in [5, p. 36]. □

Lemma 5.2. *For $k \geq 2$ and $m \geq 0$ both integers, we have*

$$k^{k-2-m}(k-1)^m \sum_{i=0}^{m} \left(\frac{k}{k-1}\right)^i (k+i) = k^{k-1}(m+1). \quad (5.2)$$

Proof. We set $\theta = k/(k-1)$, and we re-write the left-hand side of equation (5.2) as

$$k^{k-2-m}(k-1)^m \left(k \sum_{i=0}^{m} \theta^i + \sum_{i=0}^{m} i\theta^i\right). \quad (5.3)$$

Thanks to Lemma 5.1, this becomes

$$k^{k-2-m}(k-1)^m \left(k\frac{\theta^{m+1} - 1}{\theta - 1} + \frac{\theta(1-\theta^m)}{(\theta-1)^2} + \frac{m\theta^{m+1}}{\theta - 1}\right). \quad (5.4)$$

Since we had defined $\theta = k/(k-1)$, then $\theta - 1$ is equal to $1/(k-1)$ and so $1/(\theta - 1) = k - 1$. When we factor this out from the three terms inside the parentheses, then equation (5.4) becomes

$$k^{k-2-m}(k-1)^{m+1}\left(k(\theta^{m+1}-1)+\frac{\theta(1-\theta^m)}{\theta-1}+m\theta^{m+1}\right). \tag{5.5}$$

Now, the $\theta/(\theta-1)$ in the middle term can be replaced with k, so after simplifying we get

$$k^{k-2-m}(k-1)^{m+1}\Big(k(\theta^{m+1}-\theta^m)+m\theta^{m+1}\Big). \tag{5.6}$$

We now factor out $\theta^m = k^m/(k-1)^m$ and simplify to get

$$k^{k-2}(k-1)\Big(k(\theta-1)+m\theta\Big). \tag{5.7}$$

Once again we replace $(\theta-1)$ with $1/(k-1)$ and θ with $k/(k-1)$, and after simplifying we get

$$k^{k-2}\Big(k+mk\Big) = k^{k-1}(1+m), \tag{5.8}$$

as desired. □

6. Proof of Theorem 4.1

We now have all the pieces we need to prove our main result.

Proof of Theorem 4.1. For convenience, we will label the three parts of equation (4.2) as follows:

$$A_n = \left(k^k + (k-1)^{k-1}\right)\sum_{i=0}^{n}\mathcal{R}_i\mathcal{R}_{n-i}, \tag{6.1}$$

$$B_n = k^{k-1}(n+k-2)\mathcal{R}_{n+k-1}, \tag{6.2}$$

$$C_n = \sum_{j=0}^{k-2}\left(k^j\cdot(k-1)^{k-2-j}\right)(n+k+j-1)\mathcal{R}_{n+j}. \tag{6.3}$$

To show that $A_n = B_n - C_n$, we will show that their generating functions, which we will write as $A(x), B(x)$, and $C(x)$, satisfy

$$A(x) = B(x) - C(x). \tag{6.4}$$

Generating function for the first part. We begin with $A(x)$. Since the generating function for $\mathcal{R}_n$ is

$$\sum_{n=0}^{\infty}\mathcal{R}_n x^n = \frac{x}{1-x-x^k}, \tag{6.5}$$

then the Cauchy product rule [11, p. 36] tells us that the generating function for the self-convolution $\sum_{i=0}^{n} \mathcal{R}_i \mathcal{R}_{n-i}$ will be give by

$$\sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \mathcal{R}_i \mathcal{R}_{n-i} \right) x^n = \left(\frac{x}{1-x-x^k} \right)^2 = \frac{x^2}{(1-x-x^k)^2}.$$

So, our generating function $A(x)$ for the sequence A_n from equation (6.1) is

$$A(x) = \left(k^k + (k-1)^{k-1}\right) \frac{x^2}{(1-x-x^k)^2}. \tag{6.6}$$

Generating function for the second part. Next, we look at $B(x)$. From the definition of B_n in equation (6.2) we have that its generating function $B(x)$ is

$$B(x) = k^{k-1} \sum_{n=0}^{\infty} (n+k-2) \mathcal{R}_{n+k-1} x^n. \tag{6.7}$$

We notice that this is a term-by-term derivative of another, simpler power series, as shown here:

$$B(x) = k^{k-1} \sum_{n=0}^{\infty} \mathcal{R}_{n+k-1} \cdot \frac{1}{x^{k-3}} \frac{d}{dx} x^{n+k-2}. \tag{6.8}$$

We re-arrange the terms in this sum to give us

$$B(x) = \frac{k^{k-1}}{x^{k-3}} \cdot \frac{d}{dx} \sum_{n=0}^{\infty} \mathcal{R}_{n+k-1} x^{n+k-2}. \tag{6.9}$$

This is almost, but not quite, what we want, because the subscript for $\mathcal{R}_{n+k-1}$ is not quite a perfect match for the exponent in x^{n+k-2}. An easy fix is to multiply and divide by x, giving us

$$B(x) = \frac{k^{k-1}}{x^{k-3}} \cdot \frac{d}{dx} \left(\frac{1}{x} \sum_{n=0}^{\infty} \mathcal{R}_{n+k-1} x^{n+k-1} \right). \tag{6.10}$$

We now notice that the sum in the above equation starts at $n = 0$ with $\mathcal{R}_{k-1} x^{k-1}$ and so if we re-index the sum to start with $\mathcal{R}_0 x^0$ and then subtract the unwanted terms from $\mathcal{R}_0 x^0$ up to $\mathcal{R}_{k-2} x^{k-2}$, we have

$$B(x) = \frac{k^{k-1}}{x^{k-3}} \cdot \frac{d}{dx} \left(\frac{1}{x} \sum_{n=0}^{\infty} \mathcal{R}_n x^n - \frac{1}{x} \sum_{i=0}^{k-2} \mathcal{R}_i x^i \right). \tag{6.11}$$

The first sum is simply the generating function for $\mathcal{R}_n$ as seen in equation (6.5), and for the second sum we note from equation (4.1) that $\mathcal{R}_0 = 0$ and $\mathcal{R}_i = 1$ for i between

1 and k, and so we now have

$$B(x) = \frac{k^{k-1}}{x^{k-3}} \cdot \frac{d}{dx} \left(\frac{1}{1-x-x^k} - \frac{1}{x} \sum_{i=1}^{k-2} x^i \right). \tag{6.12}$$

We now distribute the $1/x$ into the sum on the right of the above equation, giving us (after re-indexing)

$$B(x) = \frac{k^{k-1}}{x^{k-3}} \cdot \frac{d}{dx} \left(\frac{1}{1-x-x^k} - \sum_{i=0}^{k-3} x^i \right). \tag{6.13}$$

When we apply the derivative to the right-hand side, we get

$$B(x) = \frac{k^{k-1}}{x^{k-3}} \left(\frac{1+kx^{k-1}}{(1-x-x^k)^2} - \sum_{i=0}^{k-3} ix^{i-1} \right). \tag{6.14}$$

and after distributing, we get

$$B(x) = \frac{k^{k-1}+k^k x^{k-1}}{x^{k-3}(1-x-x^k)^2} - \frac{k^{k-1}}{x^{k-3}} \sum_{i=0}^{k-3} ix^{i-1}. \tag{6.15}$$

Generating function for the third part. Finally, we turn our attention to $C(x)$. From the definition of C_n in equation (6.3) we have that its generating function $C(x)$ is

$$C(x) = \sum_{n=0}^{\infty} \sum_{j=0}^{k-2} \left(k^j \cdot (k-1)^{k-2-j}\right)(n+k+j-1)\mathcal{R}_{n+j}x^n. \tag{6.16}$$

When we switch the order of summation, we have

$$C(x) = \sum_{j=0}^{k-2} \left(k^j \cdot (k-1)^{k-2-j}\right) \sum_{n=0}^{\infty} (n+k+j-1)\mathcal{R}_{n+j}x^n, \tag{6.17}$$

and just as with $B(x)$ earlier we notice that the power series on the right is a term-by-term derivative of another, simpler power series, as given here:

$$C(x) = \sum_{j=0}^{k-2} \left(k^j \cdot (k-1)^{k-2-j}\right) \sum_{n=0}^{\infty} \frac{1}{x^{k+j-2}} \frac{d}{dx} \mathcal{R}_{n+j}x^{n+k+j-1}. \tag{6.18}$$

We re-arrange the terms to give us

$$C(x) = \sum_{j=0}^{k-2} \frac{k^j \cdot (k-1)^{k-2-j}}{x^{k+j-2}} \cdot \frac{d}{dx} \sum_{n=0}^{\infty} \mathcal{R}_{n+j}x^{n+k+j-1}. \tag{6.19}$$

Once again we notice that this is almost, but not quite, what we want, because the subscript for $\mathcal{R}_{n+j}$ is not quite a perfect match for the exponent in $x^{n+k+j-1}$. An easy fix is to pull out x^{k-1}, giving us

$$C(x) = \sum_{j=0}^{k-2} \frac{k^j \cdot (k-1)^{k-2-j}}{x^{k+j-2}} \cdot \frac{d}{dx} x^{k-1} \sum_{n=0}^{\infty} \mathcal{R}_{n+j} x^{n+j}. \tag{6.20}$$

We note that each sum on the right of the above equation starts at $n = 0$ with $\mathcal{R}_j x^j$, and so if we re-index each sum to start with $\mathcal{R}_0 x^0$ and then subtract the unwanted terms from $\mathcal{R}_0 x^0$ up to $\mathcal{R}_{j-1} x^{j-1}$, we have

$$C(x) = \sum_{j=0}^{k-2} \frac{k^j \cdot (k-1)^{k-2-j}}{x^{k+j-2}} \cdot \frac{d}{dx} \left(x^{k-1} \sum_{n=0}^{\infty} \mathcal{R}_n x^n - x^{k-1} \sum_{i=0}^{j-1} \mathcal{R}_i x^i \right), \tag{6.21}$$

with the understanding that when $j = 0$ the last sum on the right is an empty (zero) sum.

Now, the first sum inside the parentheses on the right of equation (6.21) is simply the generating function for $\mathcal{R}_n$ as seen in equation (6.5), and for the second sum we recall once again from equation (4.1) that $\mathcal{R}_0 = 0$ and $\mathcal{R}_i = 1$ for i between 1 and k. This means that our last sum on the right of equation (6.21) actually starts at $i = 1$ (because, again, $\mathcal{R}_0 = 0$ by definition) and has just powers of x without coefficients. Putting this all together, we see that equation (6.21) becomes

$$C(x) = \sum_{j=0}^{k-2} \frac{k^j \cdot (k-1)^{k-2-j}}{x^{k+j-2}} \cdot \frac{d}{dx} \left(\frac{x^k}{1-x-x^k} - x^{k-1} \sum_{i=1}^{j-1} x^i \right). \tag{6.22}$$

We note that the last sum on the right is an empty sum (and hence is zero) when $j = 0$ or $j = 1$. We now put the x^{k-1} back into the last sum on the right, giving us

$$C(x) = \sum_{j=0}^{k-2} \frac{k^j \cdot (k-1)^{k-2-j}}{x^{k+j-2}} \cdot \frac{d}{dx} \left(\frac{x^k}{1-x-x^k} - \sum_{i=1}^{j-1} x^{k+i-1} \right). \tag{6.23}$$

When we take the derivative and simplify, we have

$$C(x) = \sum_{j=0}^{k-2} \frac{k^j \cdot (k-1)^{k-2-j}}{x^{k+j-2}} \left(\frac{kx^{k-1} - (k-1)x^k}{(1-x-x^k)^2} - \sum_{i=1}^{j-1} (k+i-1) x^{k+i-2} \right).$$

After canceling the common x^{k-1} term everywhere, we have

$$C(x) = \sum_{j=0}^{k-2} \frac{k^j \cdot (k-1)^{k-2-j}}{x^{j-1}} \left(\frac{k-(k-1)x}{(1-x-x^k)^2} - \sum_{i=1}^{j-1} (k+i-1) x^{i-1} \right). \tag{6.24}$$

We re-index that last sum by replacing i with $i+1$, giving us

$$C(x) = \sum_{j=0}^{k-2} \frac{k^j \cdot (k-1)^{k-2-j}}{x^{j-1}} \left(\frac{k-(k-1)x}{(1-x-x^k)^2} - \sum_{i=0}^{j-2} (k+i)x^i \right). \tag{6.25}$$

After distributing and re-organizing, we have

$$C(x) = \frac{kx-(k-1)x^2}{(1-x-x^k)^2} \sum_{j=0}^{k-2} \left(\frac{k}{x}\right)^j (k-1)^{k-2-j} - x \sum_{j=0}^{k-2} \left(\frac{k}{x}\right)^j (k-1)^{k-2-j} \sum_{i=0}^{j-2} (k+i)x^i. \tag{6.26}$$

For the first sum in the above equation, we use the identity

$$\sum_{j=0}^{m} a^j b^{m-j} = \frac{a^{m+1}-b^{m+1}}{a-b} \tag{6.27}$$

to write

$$\sum_{j=0}^{k-2} \left(\frac{k}{x}\right)^j (k-1)^{k-2-j} = \frac{(k/x)^{k-1}-(k-1)^{k-1}}{k/x-(k-1)} = \frac{x^2\left((k/x)^{k-1}-(k-1)^{k-1}\right)}{kx-(k-1)x^2},$$

and when we substitute this into equation (6.26) and simplify, we get

$$C(x) = \frac{x^2\left((k/x)^{k-1}-(k-1)^{k-1}\right)}{(1-x-x^k)^2} - x \sum_{j=0}^{k-2} \left(\frac{k}{x}\right)^j (k-1)^{k-2-j} \sum_{i=0}^{j-2} (k+i)x^i. \tag{6.28}$$

We now multiply the top and bottom of the first term by x^{k-3} to give us

$$C(x) = \frac{k^{k-1}-(k-1)^{k-1}x^{k-1}}{x^{k-3}(1-x-x^k)^2} - x \sum_{j=0}^{k-2} \left(\frac{k}{x}\right)^j (k-1)^{k-2-j} \sum_{i=0}^{j-2} (k+i)x^i. \tag{6.29}$$

We now turn our attention to the double sum on the right of equation (6.29). We move the x^j term inside the second sum, we bring that x^i term into the denominator, we note that the first sum can start at $j=2$ instead of at $j=0$, and then we bring out the second summation, giving us

$$C(x) = \frac{k^{k-1}-(k-1)^{k-1}x^{k-1}}{x^{k-3}(1-x-x^k)^2} - x \sum_{j=2}^{k-2} \sum_{i=0}^{j-2} k^j (k-1)^{k-2-j} \frac{(k+i)}{x^{j-i}}. \tag{6.30}$$

We wish to re-index this double sum, and so let us first describe the values for i and j that appear in the double sum:

$$\begin{aligned} j=2: &\quad i=0.\\ j=3: &\quad i=0,1.\\ j=4: &\quad i=0,1,2.\\ j=5: &\quad i=0,1,2,3.\\ &\vdots\\ j=k-2: &\quad i=0,1,2,3,\ldots,k-4. \end{aligned}$$

If we now let $w = j - i$, we see that w runs from $w = 2$ (which covers the pairs $(j = 1, i = 0)$ and $(j = 2, i = 1)$ and $(j = 3, i = 2)$, and so on) up to $w = k - 2$ which covers only the pair $(j = k - 2, i = 0)$. So, we can cover all of this by letting i run from $i = 0$ to $i = k - 2 - w$, and so when we re-write our double sum in terms of w and i, we have

$$C(x) = \frac{k^{k-1} - (k-1)^{k-1}x^{k-1}}{x^{k-3}(1-x-x^k)^2} - x\sum_{w=2}^{k-2}\sum_{i=0}^{k-2-w} k^{w+i}(k-1)^{k-2-w-i}\frac{(k+i)}{x^w}. \qquad (6.31)$$

When we pull out some terms from the inner sum, we have

$$C(x) = \frac{k^{k-1} - (k-1)^{k-1}x^{k-1}}{x^{k-3}(1-x-x^k)^2} - x\sum_{w=2}^{k-2}\frac{k^w(k-1)^{k-2-w}}{x^w}\sum_{i=0}^{k-2-w}\left(\frac{k}{k-1}\right)^i(k+i). \qquad (6.32)$$

We re-index once more, letting $m = k - 2 - w$ so that m runs from 0 to $k - 4$, giving us

$$C(x) = \frac{k^{k-1} - (k-1)^{k-1}x^{k-1}}{x^{k-3}(1-x-x^k)^2} - x\sum_{m=0}^{k-4}\frac{k^{k-2-m}(k-1)^m}{x^{k-2-m}}\sum_{i=0}^{m}\left(\frac{k}{k-1}\right)^i(k+i). \qquad (6.33)$$

We now apply Lemma 5.2, giving us

$$C(x) = \frac{k^{k-1} - (k-1)^{k-1}x^{k-1}}{x^{k-3}(1-x-x^k)^2} - x\sum_{m=0}^{k-4}\frac{1}{x^{k-2-m}}k^{k-1}(m+1). \qquad (6.34)$$

Cleaning up a bit, this gives us

$$C(x) = \frac{k^{k-1} - (k-1)^{k-1}x^{k-1}}{x^{k-3}(1-x-x^k)^2} - \frac{k^{k-1}}{x^{k-3}}\sum_{m=0}^{k-4}(m+1)x^m. \qquad (6.35)$$

That last sum is empty for $k < 4$, but otherwise it starts with $1 + 2x + 3x^2$ and ends with $(k-3)x^{k-4}$, so we can re-index it by $i = m + 1$ and write it as

$$C(x) = \frac{k^{k-1} - (k-1)^{k-1}x^{k-1}}{x^{k-3}(1-x-x^k)^2} - \frac{k^{k-1}}{x^{k-3}}\sum_{i=0}^{k-3} ix^{i-1}. \qquad (6.36)$$

Conclusion of proof. When we compare our formula for $C(x)$ in (6.36) with our formula for $B(x)$ in (6.15), we note that

$$B(x)-C(x)=\frac{k^k x^{k-1}}{x^{k-3}(1-x-x^k)^2}-\frac{-(k-1)^{k-1}x^{k-1}}{x^{k-3}(1-x-x^k)^2}=\frac{\left(k^k+(k-1)^{k-1}\right)x^2}{(1-x-x^k)^2},$$

which is a perfect match for $A(x)$ in equation (6.6), as desired. □

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